

CONSTRUCTION OF THE PATHS OF BROWNIAN MOTIONS ON STAR GRAPHS

VADIM KOSTRYKIN, JÜRGEN POTTHOFF, AND ROBERT SCHRADER

ABSTRACT. Pathwise constructions of Brownian motions which satisfy all possible boundary conditions at the vertex of star graphs are given.

1. INTRODUCTION AND PRELIMINARIES

In the recent years there was a growing interest in *metric graphs* because of their wide range of important applications, see, e.g., the articles in [10] and the references given there. The simplest metric graphs are *star* or *single vertex graphs*: They can be defined as a set having a finite collection of subsets isomorphic to \mathbb{R}_+ , called *external edges*, where the points corresponding to the origin of \mathbb{R}_+ under these isomorphisms are identified and form the *vertex* of the graph. One may visualize a star graph as a finite number of rays in \mathbb{R}^2 going out from the origin.

On the other hand, in his pioneering articles [11–13] Feller investigated the problem of characterization and construction of all Brownian motions on intervals. This problem found a complete solution in the work of Itô and McKean [19, 20]. In particular, in [19] Itô and McKean solved the problem of construction of the paths of all Brownian motions on the semi-line \mathbb{R}_+ employing the theory of the local time of Brownian motion [30], and the theory of (strong) Markov processes [3, 6–8, 16].

Therefore it is natural to investigate Feller’s problem on metric graphs, and in particular on star graphs. In fact, the Walsh process introduced by Walsh in [38] as a generalization of the skew Brownian motion [20] is the most basic example of a Brownian motion on a star graph, and in the present article it will serve — together with its local time at the vertex — as the main building block of our constructions. The Brownian motions constructed here on star graphs are then the basic building blocks of Brownian motions on general, finite metric graphs in the article [24] of the present authors.

The main ideas for the construction of Brownian motions with boundary conditions at the vertex compatible with Feller’s theorem (see below, theorem 1.5) are those which can be found in the above mentioned work by Itô and McKean (cf. also [23, Chapter 6]): The reflecting Brownian motion in the case of \mathbb{R}_+ is replaced by a Walsh process [38] (cf. also, e.g., [2]) on the single vertex graph, and then slowing down

Date: February 22, 2011.

2010 Mathematics Subject Classification. 60J65, 60J45, 60H99, 58J65, 35K05, 05C99.

Key words and phrases. Metric graphs, Brownian motion, Feller processes, Feller’s theorem.

and the killing of this process on the scale of its local time at the vertex are used to construct processes implementing the various forms of the Wentzell boundary condition. We provide a number of arguments which — at least on a technical level — are rather different from those found in the standard literature. For example, whenever possible, we use arguments based on Dynkin's formula to derive the domain of the generator (i.e., the boundary conditions at the vertex). This approach appears to be much simpler and more intuitive than the one with standard arguments [19, 20, 23] for the semi-line \mathbb{R}_+ , which is based on rather tricky calculation of heat kernels with the help of Lévy's theorem. Moreover, for the case of killing, instead of using the standard first passage time formula for the hitting time of the vertex we use a first passage time formula for the lifetime of the process. In the opinion of the authors this leads to much simpler computations of the transition kernels than those in [19, 20, 23] for a Brownian motion on \mathbb{R}_+ . In all cases we also derive explicit expressions of the analogues of the *quantum mechanical scattering matrix* on star graphs.

The article is organized as follows. In several subsections of the present section we set up our notation and discuss some preparatory results. In section 2 we recall the construction of a Walsh process on a metric graph. In section 3 we construct a Walsh process on the single vertex graph with an elastic boundary condition at the vertex, while in section 4 we construct a Walsh process with a sticky boundary condition at the vertex. The most general Brownian motion on a star graph is obtained in section 5.

1.1. Brownian on a Star Graph and Feller's Theorem. From now we shall consider a fixed star graph \mathcal{G} with vertex v and $n \in \mathbb{N}$ external edges l_1, \dots, l_n . \mathcal{G} is equipped with the natural metric d which is induced by the metric that each external edge inherits from being isomorphic to \mathbb{R}_+ . Thus (\mathcal{G}, d) is a locally compact, complete metric space, and we shall always consider \mathcal{G} as equipped with its Borel σ -algebra. \mathcal{G}° denotes the set $\mathcal{G} \setminus \{v\}$ which we also call — by abuse of language — the *open interior* of \mathcal{G} . It is the disjoint union of n copies l_k° , $k = 1, \dots, n$, of the interval $(0, +\infty)$. Every $\xi \in \mathcal{G}^\circ$ is in one-to-one correspondence with its *local coordinates* (k, x) , where $k \in \{1, \dots, n\}$ is the index of the edge ξ belongs to, and $x > 0$ denotes the distance of ξ to v . For simplicity we shall often write $\xi = (k, x)$.

The following definition of a Brownian motion on \mathcal{G} is the generalization of the definition of a Brownian motion on the half line \mathbb{R}_+ as given by Knight in [23, Chapter 6].

Definition 1.1. A Brownian motion $X = (X_t, t \in \mathbb{R}_+)$ on \mathcal{G} is a diffusion process on \mathcal{G} , such that X with absorption at v is equivalent to a Brownian motion on the half line \mathbb{R}_+ with absorption at the origin.

Remarks 1.2. By a diffusion process we mean a strong Markov process (e.g., in the sense of [4]), which a.s. has càdlàg paths and a.s. the paths are continuous on $[0, \zeta)$, where ζ is its lifetime. Moreover, in definition 1.1 we have — as we shall usually do without any danger of confusion — identified the external leg l_k , $k = 1, \dots, n$, on which the process starts with the corresponding copy of \mathbb{R}_+ . Throughout we shall assume without loss of generality that the filtration of a Brownian motion on \mathcal{G}

satisfies the “usual conditions”, i.e., it is right continuous and complete relative to the underlying family $(P_\xi, \xi \in \mathcal{G})$ of probability measures.

$C_0(\mathcal{G})$ denotes the Banach space of continuous functions on \mathcal{G} which vanish at infinity equipped with the sup-norm.

Define $C_0^2(\mathcal{G})$ as the subspace of $C_0(\mathcal{G})$ consisting of those functions $f \in C_0(\mathcal{G})$ which are twice continuously differentiable on \mathcal{G}° , such that f'' extends from \mathcal{G}° to \mathcal{G} as a function in $C_0(\mathcal{G})$. The following lemma states some of the properties of functions in $C_0^2(\mathcal{G})$. It can be proved with the help of applications of the fundamental theorem of calculus and the mean value theorem, and the proof is omitted here.

Lemma 1.3. *Suppose that $f \in C_0^2(\mathcal{G})$, $k \in \{1, \dots, n\}$. Then the limit of $f'(\xi)$ as ξ converges to v along the open edge l_k° exists. The directional derivatives $f^{(i)}(v_k)$, $i = 1, 2$, of first and second order of f at the vertex v in direction of the edge l_k exist, and the equalities*

$$f^{(i)}(v_k) = \lim_{\xi \rightarrow v, \xi \in l_k^\circ} f^{(i)}(\xi), \quad i = 1, 2,$$

hold true. Moreover, f' vanishes at infinity.

Remark 1.4. By definition we have that for every $f \in C_0^2(\mathcal{G})$ and all $j, k = 1, \dots, n$, $f''(v_j) = f''(v_k)$, and henceforth we shall simply write $f''(v)$ for this quantity. On the other hand, in general $f'(v_j) \neq f'(v_k)$ for $j \neq k$.

It is not hard to show that every Brownian motion on a star graph is a Feller process. A convenient way to prove this is to show that its resolvent maps $C_0(\mathcal{G})$ into itself by arguments similar to those in [20, Section 3.6], and to observe that the path properties imply for all $\xi \in \mathcal{G}$, $f \in C_0(\mathcal{G})$, $P_t f(\xi)$ converges to $f(\xi)$ as t decreases to 0, where $(P_t, t \in \mathbb{R}_+)$ denotes the semigroup generated by the Brownian motion. Then one can use well-known arguments (for example, a complete proof can be found in [27]) to conclude the Feller property in its usual form, e.g., [35].

The analogue of Feller’s theorem [23, Theorem 6.2] for a Brownian motion on the single vertex graph \mathcal{G} reads as follows:

Theorem 1.5. *Assume that X is a Brownian motion on \mathcal{G} . Then there exist constants $a, b_k, c \in [0, 1]$, $k = 1, \dots, n$, with*

$$(1.1a) \quad a + c + \sum_{k=1}^n b_k = 1, \quad a \neq 1,$$

such that the domain $\mathcal{D}(A)$ of the generator A of X in $C_0(\mathcal{G})$ consists exactly of those $f \in C_0^2(\mathcal{G})$ for which

$$(1.1b) \quad af(v) + \frac{c}{2} f''(v) = \sum_{k=1}^n b_k f'(v_k)$$

holds true. Moreover, for $f \in \mathcal{D}(A)$, $Af = 1/2 f''$.

The proof in [23] for the case where \mathcal{G} has only one external edge, i.e., $\mathcal{G} = \mathbb{R}_+$, is readily modified for a general star graph \mathcal{G} . On the other hand, theorem 1.5 also follows from Feller's theorem in the case of a general metric graph [24, Theorem 1.3].

1.2. Standard Brownian Motion on the Real Line. The construction of Brownian motions on a single vertex graph with infinitesimal generator whose domain consists of functions f which satisfy the boundary conditions (1.1) is quite similar to the construction carried out for the half-line in [19], [20], [23]. This in turn is based on the properties of a standard Brownian motion on the real line, cf., e.g., [14, 15, 17, 22, 35, 40], and the works cited above. For the convenience of the reader, and for later reference, we collect the pertinent notions, tools and results here.

Let $(Q_x, x \in \mathbb{R})$ denote a family of probability measures on a measurable space (Ω', \mathcal{A}') , and let $B = (B_t, t \in \mathbb{R}_+)$ denote a standard Brownian motion defined on (Ω', \mathcal{A}') with $Q_x(B_0 = x) = 1, x \in \mathbb{R}$. It will be convenient to assume throughout that B exclusively has continuous paths. Whenever it is notationally convenient, we shall also write $B(t)$ for $B_t, t \geq 0$. Furthermore, we may suppose that there is a shift operator $\theta : \mathbb{R}_+ \times \Omega \rightarrow \Omega$, such that for all $s, t \geq 0$, $B_s \circ \theta_t = B_{s+t}$.

We shall always understand the Brownian family $(B, (Q_x, x \in \mathbb{R}))$ to be equipped with a filtration $\mathcal{J} = (\mathcal{J}_t, t \geq 0)$ which is right continuous and complete for the family $(Q_x, x \in \mathbb{R})$. (For example, \mathcal{J} could be chosen as the usual augmentation of the natural filtration of B (e.g., [22, Sect. 2.7] or [35, Sect.'s I.4, III.2]).)

For any $A \subset \mathbb{R}$, we denote by H_A^B the hitting time of A by B ,

$$(1.2) \quad H_A^B = \inf\{t > 0, B_t \in A\},$$

and we note that for all A belonging to the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ of \mathbb{R} , H_A^B is a stopping time with respect to \mathcal{J} (e.g., [35, Theorem III.2.17]). In the case where $A = \{x\}, x \in \mathbb{R}$, we also simply write H_x^B for $H_{\{x\}}^B$. We shall also denote these stopping times by $H^B(A)$ and $H^B(x)$, respectively, whenever it is typographically more convenient. The following particular cases deserve special attention. Let $x \in \mathbb{R}$. Then we have (e.g., [20, Sect. 1.7], [22, Sect. 2.6], [35, Sect.'s II.3, III.3])

$$(1.3) \quad Q_0(H_x^B \in dt) = Q_x(H_0^B \in dt) = \frac{|x|}{t} g(t, x) dt, \quad t > 0,$$

where g is the Gauß-kernel

$$(1.4) \quad g(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad t > 0, x \in \mathbb{R},$$

and

$$(1.5) \quad E_0^Q(e^{-\lambda H_x^B}) = E_x^Q(e^{-\lambda H_0^B}) = e^{-\sqrt{2\lambda}|x|}, \quad \lambda > 0.$$

Moreover, for $a < x < b$ the law of $H_{\{a,b\}}^B$ under Q_x is well-known (e.g., [20, Problem 6, Sect. 1.7]), and its expectation is given by

$$(1.6) \quad E_x^Q(H_{\{a,b\}}^B) = (x - a)(b - x).$$

Denote by $L^B = (L_t^B, t \geq 0)$ the local time of B at zero, where we choose the normalization as in, e.g., [35] (and which thus differs by a factor 2 from the one used in, e.g., [17, 22]): for $x \in \mathbb{R}$, P_x -a.s.

$$(1.7) \quad L_t^B = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda(\{s \leq t, |B_s| \leq \epsilon\}), \quad t \geq 0,$$

and here λ denotes the Lebesgue measure. Thus, in terms of its α -potential (cf. [4, Theorem V.3.13]) we have

$$(1.8) \quad u_{L^B}^\alpha(x) = E_x \left(\int_0^\infty e^{-\alpha t} dL_t^B \right) = \frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}|x|}, \quad \alpha > 0, x \in \mathbb{R},$$

which provides an efficient way to compare the various normalizations of the local time used in the literature. Slightly informally we can write

$$(1.9) \quad L_t^B = \int_0^t \delta_0(B_s) ds,$$

where δ_0 is the Dirac distribution concentrated at 0. L^B is adapted to \mathcal{J} , and non-decreasing. Moreover, for every $x \in \mathbb{R}$, P_x -a.s. the paths of L^B are continuous, and L^B is additive in the sense that

$$(1.10) \quad L_{t+s}^B = L_t^B + L_s^B \circ \theta_t, \quad s, t \in \mathbb{R}_+.$$

Similarly as above, we shall occasionally take the notational freedom to rewrite L_t^B as $L^B(t)$.

We will need the following well-known result (e.g., [20, Section 2.2, Problem 3]):

Lemma 1.6. *The joint law of $|B_t|$ and L_t , $t > 0$, under Q_0 is given by*

$$(1.11) \quad Q_0(|B_t| \in dx, L_t^B \in dy) = 2 \frac{x+y}{\sqrt{2\pi t^3}} e^{-(x+y)^2/2t} dx dy, \quad x, y \geq 0.$$

Let $K^B = (K_r^B, r \geq 0)$ denote the right continuous pseudo-inverse of L ,

$$(1.12) \quad K_r^B = \inf\{t \geq 0, L_t^B > r\}, \quad r \geq 0.$$

Note that due to the a.s. continuity of L^B we have a.s. $L_{K_r^B}^B = r$. In appendix B of [25] the present authors proved the following

Lemma 1.7. *For any $r \geq 0$*

$$(1.13) \quad Q_0(K_r^B \in dt) = \frac{r}{t} g(t, r) dt, \quad t > 0,$$

and

$$(1.14) \quad E_0^Q(e^{-\lambda K_r^B}) = e^{-\sqrt{2\lambda}r}, \quad \lambda > 0$$

holds.

Moreover, we shall make use of the following lemma, which is similar to results in Section 6.4 of [22], and which is proved in appendix B of [25], too.

Lemma 1.8. *Under Q_0 , $L^B(H_{\{-x, +x\}}^B)$, $x > 0$, is exponentially distributed with mean x .*

1.3. First Passage Time Formula for Single Vertex Graphs. In this subsection we set up some additional notation which will be used throughout this article. Also we record a special form of the well-known first passage time formula, e.g., [20, 34].

Let X be a Brownian motion on \mathcal{G} in the sense of definition 1.1 defined on a family $(\Omega, \mathcal{A}, \mathcal{F} = (\mathcal{F}_t, t \geq 0), (P_\xi, \xi \in \mathcal{G}))$ of filtered probability spaces. Let H_v be the hitting time of the vertex v . It follows from definition 1.1 that for all $\xi \in \mathcal{G}$, $P_\xi(H_v < +\infty) = 1$. For $\lambda > 0$, set

$$(1.15) \quad e_\lambda(\xi) = E_\xi(\exp(-\lambda H_v)) = e^{-\sqrt{2\lambda}d(\xi, v)}, \quad \xi \in \mathcal{G},$$

where $E_\xi(\cdot)$ denotes expectation with respect to P_ξ . The last equality follows from formula (1.5).

Recall that we denote the natural metric on \mathcal{G} by d . We introduce another symmetric map d_v from $\mathcal{G} \times \mathcal{G}$ to \mathbb{R}_+ defined by

$$(1.16) \quad d_v(\xi, \eta) = d(\xi, v) + d(v, \eta), \quad \xi, \eta \in \mathcal{G},$$

which is the “distance from ξ to η via the vertex v ”. Observe that if $\xi, \eta \in \mathcal{G}$ do not belong to the same edge, then $d_v(\xi, \eta) = d(\xi, \eta)$ holds.

Next we define two heat kernels on \mathcal{G} by

$$(1.17) \quad p(t, \xi, \eta) = \sum_{k=1}^n 1_{l_k}(\xi) g(t, d(\xi, \eta)) 1_{l_k}(\eta),$$

$$(1.18) \quad p_v(t, \xi, \eta) = \sum_{k=1}^n 1_{l_k}(\xi) g(t, d_v(\xi, \eta)) 1_{l_k}(\eta),$$

with $t > 0$, $\xi, \eta \in \mathcal{G}$. g is the Gauß-kernel defined in equation (1.4). Hence, in local coordinates $\xi = (k, x)$, $\eta = (m, y)$, $x, y \geq 0$, $k, m \in \{1, 2, \dots, n\}$, these kernels read

$$(1.19) \quad p(t, (k, x), (m, y)) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \delta_{km}$$

$$(1.20) \quad p_v(t, (k, x), (m, y)) = \frac{1}{\sqrt{2\pi t}} e^{-(x+y)^2/2t} \delta_{km}.$$

The *Dirichlet heat kernel* p^D on \mathcal{G} is then given by

$$(1.21) \quad p^D(t, \xi, \eta) = p(t, \xi, \eta) - p_v(t, \xi, \eta), \quad t > 0, \xi, \eta \in \mathcal{G}.$$

It is the transition density of a strong Markov process with state space $\mathcal{G}^\circ \cup \{\Delta\}$ which on every edge of \mathcal{G}° is equivalent to a Brownian motion until the moment of reaching the vertex when it is killed, and Δ denotes a universal cemetery state for all stochastic processes considered adjoined to \mathcal{G} as an isolated point. (Observe that this process is *not* a Brownian motion on \mathcal{G} in the sense of definition 1.1.)

The *Dirichlet resolvent* $R^D = (R_\lambda^D, \lambda > 0)$ on \mathcal{G} is defined by

$$(1.22) \quad R_\lambda^D f(\xi) = E_\xi \left(\int_0^{H_v} e^{-\lambda t} f(X_t) dt \right), \quad \lambda > 0, \xi \in \mathcal{G}, f \in B(\mathcal{G}).$$

It is easy to see that R_λ^D has the following integral kernel on \mathcal{G}

$$(1.23) \quad r_\lambda^D(\xi, \eta) = r_\lambda(\xi, \eta) - r_{v,\lambda}(\xi, \eta), \quad \xi, \eta \in \mathcal{G},$$

where for $\xi, \eta \in \mathcal{G}$,

$$(1.24) \quad r_\lambda(\xi, \eta) = \sum_{k=1}^n 1_{l_k}(\xi) \frac{e^{-\sqrt{2\lambda} d(\xi, \eta)}}{\sqrt{2\lambda}} 1_{l_k}(\eta),$$

and

$$(1.25) \quad \begin{aligned} r_{v,\lambda}(\xi, \eta) &= \sum_{k=1}^n 1_{l_k}(\xi) \frac{e^{-\sqrt{2\lambda} d_v(\xi, \eta)}}{\sqrt{2\lambda}} 1_{l_k}(\eta) \\ &= \sum_{k=1}^n 1_{l_k}(\xi) \frac{1}{\sqrt{2\lambda}} e_\lambda(\xi) e_\lambda(\eta) 1_{l_k}(\eta). \end{aligned}$$

In particular, r_λ^D is the Laplace transform of the Dirichlet heat kernel (1.21) at $\lambda > 0$.

In the present context the well-known first passage time formula, e.g., [20, 34], reads as follows

$$R_\lambda f(\xi) = E_\xi \left(\int_0^S e^{-\lambda t} f(X_t) dt \right) + E_\xi (e^{-\lambda S} R_\lambda f(X_S)),$$

where S is any P_ξ -a.s. finite stopping time relative to \mathcal{J} . The choice $S = H_v$ gives the following result.

Lemma 1.9. *Let X be a Brownian motion on \mathcal{G} with resolvent $R = (R_\lambda, \lambda > 0)$. Then for all $\lambda > 0$, $\xi \in \mathcal{G}$, $f \in B(\mathcal{G})$,*

$$(1.26) \quad R_\lambda f(\xi) = R_\lambda^D f(\xi) + e_\lambda(\xi) R_\lambda f(v)$$

holds true.

The following notation will be convenient. For real valued measurable functions f, g on \mathcal{G} , with restrictions $f_k, g_k, k \in \{1, 2, \dots, n\}$, to the edges l_k we set

$$(f, g) = \int_{\mathcal{G}} f(\xi) g(\xi) d\xi = \sum_{k=1}^n (f_k, g_k),$$

where the integration is with respect to the Lebesgue measure on \mathcal{G} , and

$$(f_k, g_k) = \int_0^\infty f_k(x) g_k(x) dx,$$

whenever the integrals exist.

Assume that $f \in C_0(\mathcal{G})$. Then for $\lambda > 0$, $R_\lambda f$ belongs to the domain of the generator of X , and therefore to $C_0^2(\mathcal{G})$ (cf. subsection 1.1). It is straightforward to compute the derivative of the right hand side of formula (1.26), and we obtain the

Corollary 1.10. *For every Brownian motion X on \mathcal{G} with resolvent $R = (R_\lambda, \lambda > 0)$, and all $f \in C_0(\mathcal{G})$,*

$$(1.27) \quad (R_\lambda f)'(v_k) = 2(e_{\lambda,k}, f_k) - \sqrt{2\lambda} R_\lambda f(v), \quad k \in \{1, 2, \dots, n\},$$

holds true.

1.4. The Case $b = 0$. The case, where all parameters $b_k, k = 1, \dots, n$, in equation (1.1) vanish, is trivial in the sense that the associated Brownian motion can be constructed by a stochastic process living only on the edge where it started, and therefore it is just a classical Brownian motion on \mathbb{R}_+ in the sense of [23, Section 6.1]. This case is also discussed briefly in [23], but for the sake of completeness we include it here in somewhat more detail than in [23].

Consider a standard Brownian motion on \mathbb{R} as before, and without loss of generality assume in addition that the underlying sample space is large enough such that all constant paths in \mathbb{R} can be realized as paths of the Brownian motion. Construct from the Brownian motion a new process by stopping it when it reaches the origin of \mathbb{R} , and then kill it after an exponential holding time (independent of the Brownian motion) with rate $\beta \geq 0$. We shall only consider starting points $x \in \mathbb{R}_+$. If $\beta = 0$, then the process is simply a Brownian motion with absorption at the origin. For example, it follows from Theorem 10.1 and Theorem 10.2 in [7] that for every $\beta \geq 0$ this process is a strong Markov process, and obviously it has the path properties which make it a Brownian motion on \mathbb{R}_+ in the sense of [23, Section 6.1]. Thus, if $\xi \in \mathcal{G}, \xi \in l_k, k = 1, \dots, n$, then we just have to map this process with the isomorphisms between the edges $l_k, k = 1, \dots, n$, and the interval $[0, +\infty)$ into \mathcal{G} to obtain a Brownian motion on \mathcal{G} with start in ξ , such that it is stopped when reaching the vertex, and then is killed there after an exponential holding time with rate $\beta \geq 0$.

Let $U^0 = (U_t^0, t \geq 0)$ denote the semigroup associated with this process. It is obvious that for $f \in C_0(\mathcal{G})$ we get $U_t^0 f(v) = \exp(-\beta t) f(v), t \geq 0$. Thus for the corresponding resolvent $R^0 = (R_\lambda^0, \lambda > 0)$, and $f \in C_0(\mathcal{G})$ one finds

$$(1.28) \quad \lambda R_\lambda^0 f(v) - f(v) + \beta R_\lambda^0 f(v) = 0, \quad \lambda > 0.$$

Let A^0 be the generator of this process, and recall from theorem 1.5, that for all $f \in \mathcal{D}(A^0)$, $A^0 f(\xi) = 1/2 f''(\xi), \xi \in \mathcal{G}$. But then the identity $\lambda R_\lambda^0 = A^0 R_\lambda^0 + \text{id}$ implies the following formula

$$\frac{1}{2} (R_\lambda^0 f)''(v) + \beta R_\lambda^0 f(v) = 0.$$

For every $\lambda > 0$ R_λ^0 maps $C_0(\mathcal{G})$ onto $\mathcal{D}(A^0)$. With the choice $a = (1 + \beta)^{-1} \beta, c = (1 + \beta)^{-1}$ this shows that the process realizes the boundary conditions of equations (1.1) with $b_k = 0, k = 1, \dots, n$.

Moreover, we can now use equation (1.26) combined with formula (1.28), to obtain the following explicit expression for the resolvent with $f \in C_0(\mathcal{G}), \lambda > 0$:

$$(1.29) \quad R_\lambda^0 f(\xi) = R_\lambda^D f(\xi) + \frac{1}{\beta + \lambda} e^{-\sqrt{2\lambda} d(\xi, v)} f(v), \quad \xi \in \mathcal{G},$$

where, as before, R_λ^D is the Dirichlet resolvent.

In order to compute the heat kernel associated with this process on \mathcal{G} , we invert the Laplace transforms in equation (1.29). For the first term on the right hand side this is trivial, and gives the Dirichlet heat kernel p^D , cf. equation (1.21). The second term could be handled by a formula which can be found in the tables (e.g., [9, eq. (5.6.10)]). But this formula involves the complementary error function erfc at complex arguments, and does not yield a very intuitive expression. Instead, we can simply use the observation that $t \mapsto \exp(-\beta t)$ is the inverse Laplace transform of $\lambda \mapsto (\beta + \lambda)^{-1}$. Moreover, the well-known formula for the density of the hitting time of the origin by a Brownian motion on the real line (e.g., [20, p. 25], [22, p. 96], [35, p. 102]) provides us with the following expression for the density of the first hitting time of the vertex

$$(1.30) \quad P_\xi(H_v \in ds) = \frac{d(\xi, v)}{\sqrt{2\pi s^3}} e^{-d(\xi, v)^2/2s} ds, \quad s \geq 0.$$

Using the well-known Laplace transform (e.g., [9, eq. (4.5.28)])

$$(1.31) \quad \int_0^\infty e^{-\lambda s} \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} ds = e^{-\sqrt{2\lambda}a}, \quad a > 0, \lambda > 0,$$

we infer that the inverse Laplace transform of the exponential in (1.29) is given by $P_\xi(H_v \in ds)$. Thus we obtain the following heat kernel

$$(1.32) \quad \begin{aligned} p^0(t, \xi, d\eta) &= p^D(t, \xi, \eta) d\eta - \left(\int_0^t e^{-\beta(t-s)} P_\xi(H_v \in ds) \right) \epsilon_v(d\eta), \\ &= p^D(t, \xi, \eta) d\eta - \left(\int_0^t e^{-\beta(t-s)} \frac{d(\xi, v)}{\sqrt{2\pi s^3}} e^{-d(\xi, v)^2/2s} ds \right) \epsilon_v(d\eta), \end{aligned}$$

with $\xi, \eta \in \mathcal{G}$, $t > 0$, and ϵ_v is the Dirac measure at the vertex v .

1.5. Killing via the Local Time at the Vertex. We recall from remark 1.2, that we may and will consider every Brownian motion X on \mathcal{G} with respect to a filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ which is right continuous and complete relative to $(P_\xi, \xi \in \mathcal{G})$, and such that X is strongly Markovian with respect to \mathcal{F} .

In this subsection we suppose that X is a Brownian motion on the single vertex graph \mathcal{G} with infinite lifetime, and such that the vertex is not absorbing. This entails (e.g., [35, Proposition II.2.19]) that X leaves the vertex immediately and begins a standard Brownian excursion into one of the edges. Therefore we get in this case for the hitting time H_v of the vertex $P_v(H_v = 0) = 1$, i.e., v is regular for $\{v\}$ in the sense of [4]. Consequently X has a local time $L = (L_t, t \geq 0)$ at the vertex (e.g., [4, Theorem V.3.13]). Without loss of generality, we suppose throughout this subsection that L is a *perfect continuous homogeneous additive functional (PCHAF)* of X in the sense of [40, Section III.32]. That is, L is a non-decreasing process, which is adapted to \mathcal{F} , and such that it is a.s. continuous, additive, i.e., $L_{t+s} = L_t + L_s \circ \theta_t$, and for all $\xi \in \mathcal{G}$, $P_\xi(L_0 = 0) = 1$ holds true. Moreover we may and will assume from now on that X and L are *pathwise* continuous.

Killing X exponentially on the scale of L , we can construct a new Brownian motion \hat{X} on \mathcal{G} . We shall do this using the method of [22, 23].

Let $K = (K_s, s \in \mathbb{R}_+)$ denote the right continuous pseudo-inverse of L :

$$(1.33) \quad K_s = \inf\{t \geq 0, L_t > s\}, \quad s \in \mathbb{R}_+,$$

where — as usual — we make the convention that $\inf \emptyset = +\infty$. The continuity of L entails that for every $s \in \mathbb{R}_+$, $L_{K_s} = s$. Clearly, K is increasing, and due to its right continuity it is a measurable stochastic process. Fix $s \in \mathbb{R}_+$. It is straightforward to check that for every $t \in \mathbb{R}_+$,

$$(1.34) \quad \{K_s < t\} = \{L_t > s\}.$$

Because L is adapted, the set on the right hand side belongs to \mathcal{F}_t , and since \mathcal{F} is right continuous, equality (1.34) shows that for every $s \in \mathbb{R}_+$, K_s is a stopping time relative to \mathcal{F} . We remark that since L only increases when X is at the vertex v , the continuity of X implies that for every $s \in \mathbb{R}_+$ we get $X(K_s) = v$ on $\{K_s < +\infty\}$. On the other hand, we shall argue below that L a.s. increases to $+\infty$, so that we get $X(K_s) = v$ a.s. for all $s \in \mathbb{R}_+$.

Let $\beta > 0$. Bring in the additional probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$, where P_β is the exponential law with rate β . Let S denote the associated coordinate random variable $S(s) = s$, $s \in \mathbb{R}_+$. Define

$$\hat{\Omega} = \Omega \times \mathbb{R}_+, \quad \hat{\mathcal{A}} = \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+), \quad \hat{P}_\xi = P_\xi \otimes P_\beta, \quad \xi \in \mathcal{G}.$$

We extend X , L , K , and S in the canonical way to these enlarged probability spaces, but for simplicity keep the same notation for these quantities.

Set

$$(1.35) \quad \zeta_\beta = \inf\{t \geq 0, L_t > S\},$$

and observe that since K is measurable we may write $\zeta_\beta = K_S$. Thus as above we get $X(\zeta_\beta) = v$. Define the *killed process*

$$(1.36) \quad \hat{X}_t = \begin{cases} X_t, & t < \zeta_\beta, \\ \Delta, & t \geq \zeta_\beta. \end{cases}$$

Since this prescription for killing the process X via the (PCHAF) L is slightly different from the method used in [4, 40], we cannot directly use the results proved there to conclude that the subprocess \hat{X} of X is still a strong Markov process. However, it has been proved in [25, Appendix A] that the strong Markov property is preserved under this method of killing, i.e., \hat{X} is a strong Markov process relative to its natural filtration (actually relative to a larger filtration, but we will not use this here). Now we may employ the arguments in section 2.7 of [22], or in section III.2 of [35] to conclude that \hat{X} is a strong Markov process with respect to the universal right continuous and complete augmentation of its natural filtration.

It is clear that \hat{X} has a.s. right continuous paths which admit left limits, and that its paths on $[0, \zeta_\beta)$ are equal to those of X , and thus are continuous on this random time interval. Moreover, it is obvious that for every $\xi \in \mathcal{G}^\circ$, we have $P_\xi(\zeta_\beta \geq H_v) = 1$.

Therefore, up to its hitting time of the vertex, \hat{X} is equivalent to a Brownian motion on the edge to which ξ belongs, because so is X . Altogether we have proved that — under the hypothesis that L is a PCHAF, which will be argued below in all cases that we consider — \hat{X} is a Brownian motion on \mathcal{G} in the sense of definition 1.1.

There is a simple, useful relationship between the resolvents R and \hat{R} of the processes X and \hat{X} , respectively. Recall our convention that all functions f on \mathcal{G} are extended to $\mathcal{G} \cup \{\Delta\}$ by $f(\Delta) = 0$.

Lemma 1.11. *For all $\lambda > 0$, $f \in B(\mathcal{G})$, $\xi \in \mathcal{G}$,*

$$(1.37) \quad \hat{R}_\lambda f(\xi) = R_\lambda f(\xi) - e_\lambda(\xi) \hat{E}_v(e^{-\lambda \zeta_\beta}) R_\lambda f(v)$$

holds true, where e_λ is defined in equation (1.15).

Proof. For $\lambda > 0$, $f \in B(\mathcal{G})$, $\xi \in \mathcal{G}$

$$\begin{aligned} \hat{R}_\lambda f(\xi) &= \hat{E}_\xi \left(\int_0^{\zeta_\beta} e^{-\lambda t} f(X_t) dt \right) \\ &= R_\lambda f(\xi) - \hat{E}_\xi \left(e^{-\lambda \zeta_\beta} \int_0^\infty e^{-\lambda t} f(X_{t+\zeta_\beta}) dt \right). \end{aligned}$$

By construction, the last expectation value is equal to

$$\beta \int_0^\infty e^{-\beta s} \int_0^\infty e^{-\lambda t} E_\xi \left(e^{-\lambda K_s} f(X_{t+K_s}) \right) dt ds,$$

where we used Fubini's theorem. Consider the expectation value under the integrals, and recall that for fixed $s \in \mathbb{R}_+$, K_s is an \mathcal{F} -stopping time, while X is strongly Markovian relative to \mathcal{F} . Hence we can compute as follows

$$\begin{aligned} E_\xi \left(e^{-\lambda K_s} f(X_{t+K_s}) \right) &= E_\xi \left(e^{-\lambda K_s} E_\xi \left(f(X_{t+K_s}) \mid \mathcal{F}_{K_s} \right) \right) \\ &= E_\xi \left(e^{-\lambda K_s} E_{X(K_s)}(f(X_t)) \right) \\ &= E_\xi \left(e^{-\lambda K_s} E_v(f(X_t)) \right), \end{aligned}$$

where we used the fact that a.s. $X(K_s) = v$. So far we have established

$$\hat{R}_\lambda f(\xi) = R_\lambda f(\xi) - \hat{E}_\xi(e^{-\lambda \zeta_\beta}) R_\lambda f(v).$$

In order to compute the expectation value on the right hand side, we first remark that because L is zero until X hits the vertex for the first time, we find that for given $s \in \mathbb{R}_+$, $K_s \geq H_v$, and therefore $K_s = H_v + K_s \circ \theta_{H_v}$. Hence, and again by the strong Markov property,

$$\begin{aligned} E_\xi(e^{-\lambda K_s}) &= E_\xi(e^{-\lambda H_v} e^{-\lambda K_s \circ \theta_{H_v}}) \\ &= E_\xi(e^{-\lambda H_v} E_\xi(e^{-\lambda K_s \circ \theta_{H_v}} \mid \mathcal{F}_{H_v})) \\ &= E_\xi(e^{-\lambda H_v}) E_v(e^{-\lambda K_s}), \end{aligned}$$

Integrating the last identity against the exponential law in the variable s , we find with formula (1.15)

$$\hat{E}_\xi(e^{-\lambda\zeta_\beta}) = e_\lambda(\xi) \hat{E}_v(e^{-\lambda\zeta_\beta}),$$

and the proof is finished. \square

Remark 1.12. Formula (1.37) is quite useful, because if the resolvent of X is known, then — in view of equation (1.15) — it reduces the calculation of \hat{R}_λ to the computation of the Laplace transform of the density of ζ_β under \hat{P}_v .

2. THE WALSH PROCESS

The most basic process — which on a single vertex graph plays the same role as a reflected Brownian motion on the half line — is the well-known *Walsh process*, which we denote by $W = (W_t, t \geq 0)$. It corresponds to the case where the parameters a and c in the boundary condition (1.1) both vanish. This process has been introduced by Walsh in [38] as a generalization of the skew Brownian motion discussed in [20, Chapter 4.2] to a process in \mathbb{R}^2 which only moves on rays connected to the origin.

A pathwise construction of the Walsh process in the present context is as follows. Consider the paths of the standard Brownian motion $B = (B_t, t \geq 0)$ on \mathbb{R} , and its associated reflected Brownian motion $|B| = (|B_t|, t \geq 0)$, where $|\cdot|$ denotes absolute value. Let $Z = \{t \geq 0, B_t = 0\}$. Then its complement Z^c is open, and hence it is the pairwise disjoint union of a countable family of *excursion intervals* $I_j = (t_j, t_{j+1})$, $j \in \mathbb{N}$. Let $R = (R_j, j \in \mathbb{N})$ be an independent sequence of identically distributed random variables, independent of B , with values in $\{1, 2, \dots, n\}$ such that $R_j, j \in \mathbb{N}$, takes the value $k \in \{1, 2, \dots, n\}$ with probability $w_k \in [0, 1]$, $\sum_k w_k = 1$. Now define $W_t = v$ if $t \in Z$, and if $t \in I_j$, and $R_j = k$ set $W_t = (k, |B_t|)$. In other words, when starting at $\xi \in \mathcal{G}^\circ$, the process moves as a Brownian motion on the edge containing ξ until it hits the vertex at time H_v , and then W performs Brownian excursions from the vertex v into the edges l_k , $k \in \{1, 2, \dots, n\}$, whereby the edge l_k is selected with probability w_k .

As for the standard Brownian motion on \mathbb{R} (cf. subsection 1.2), we may and will assume without loss of generality that W has exclusively continuous paths.

Walsh has remarked in the epilogue of [38], cf. also [2], that it is not completely straightforward to prove that this stochastic process is strongly Markovian. A proof of the strong Markov property based on Itô's excursion theory [18] has been given in [36, 37]. A construction of this process via its Feller semigroup can be found in [2] (cf. also the references quoted there for other approaches).

Next we check that the Walsh process has a generator with boundary condition at the vertex given by (1.1) with $a = c = 0$. Let $f \in \mathcal{D}(A^w)$. At the vertex v Dynkin's form for the generator reads

$$(2.1) \quad A^w f(v) = \lim_{\epsilon \downarrow 0} \frac{E_v \left(f(X(H_{v,\epsilon}^w)) \right) - f(v)}{E_v(H_{v,\epsilon}^w)},$$

where $H_{v,\epsilon}^w$ is the hitting time of the complement of the open ball $B_\epsilon(v)$ of radius $\epsilon > 0$ around v .

Lemma 2.1. *For the Walsh process $E_v(H_{v,\epsilon}^w) = \epsilon^2$.*

Proof. Since by construction W has infinite lifetime, $H_{v,\epsilon}^w$ is the hitting time of the set of the n points with local coordinates (k, ϵ) , $k = 1, \dots, n$. Therefore, by the independence of the choice of the edge for the values of the excursion, it follows that under P_v the stopping time $H_{v,\epsilon}^w$ has the same law as the hitting time of the point $\epsilon > 0$ of a reflected Brownian motion on \mathbb{R}_+ , starting at 0. Thus the statement of the lemma follows from equation (1.6). \square

From the construction of W we immediately get

$$E_v\left(f(W(H_{v,\epsilon}^w))\right) = \sum_{k=1}^n w_k f_k(\epsilon),$$

with the notation $f_k(x) = f(k, x)$, $x \in \mathbb{R}_+$. Inserting this into equation (2.1) we obtain

$$A^w f(v) = \lim_{\epsilon \downarrow 0} \epsilon^{-2} \sum_{k=1}^n w_k (f_k(\epsilon) - f(v)),$$

and since $f'(v_k)$ exists (cf. lemma 1.3) it is obvious that this entails the condition

$$(2.2) \quad \sum_{k=1}^n w_k f'(v_k) = 0.$$

For later use we record this result as

Theorem 2.2. *Consider the boundary condition (1.1) with $a = c = 0$, and $b \in [0, 1]^n$. Let W be a Walsh process as constructed above with the choice $w_k = b_k$, $k \in \{1, 2, \dots, n\}$. Then the generator A^w of W is $1/2$ times the second derivative on \mathcal{G} with domain consisting of those $f \in C_0^2(\mathcal{G})$ which satisfy condition (1.1b).*

For the remainder of this section we make the choice $a = c = 0$, $w_k = b_k$, $k \in \{1, 2, \dots, n\}$ in (1.1).

Next we compute the resolvent of W . Let $\lambda > 0$, $f \in C_0(\mathcal{G})$, and consider first $\xi = v$. Without loss of generality, we may assume that W has been constructed pathwise from a standard Brownian motion B as described above, and that B is as in subsection 1.2. Then we get

$$E_v(f(W_t)) = \sum_{m=1}^n b_m E_0^Q(f_m(|B_t|)).$$

Hence we find for the resolvent R^w of the Walsh process

$$(2.3a) \quad R_\lambda^w f(v) = \int_{\mathcal{G}} r_\lambda^w(v, \eta) f(\eta) d\eta$$

with resolvent kernel $r_\lambda^w(v, \eta)$, $\eta \in \mathcal{G}$, given by

$$(2.3b) \quad r_\lambda^w(v, \eta) = \sum_{m=1}^n 2b_m \frac{e^{-\sqrt{2\lambda}d(v, \eta)}}{\sqrt{2\lambda}} 1_{l_m}(\eta),$$

and where the integration in (2.3a) is with respect to the Lebesgue measure on \mathcal{G} .

Now let $\xi \in \mathcal{G}$. We use the first passage time formula (1.26) together with formulae (1.15) and (2.3), and obtain

Lemma 2.3. *The resolvent of the Walsh process on \mathcal{G} is given by*

$$(2.4a) \quad R_\lambda^w f(\xi) = \int_{\mathcal{G}} r_\lambda^w(\xi, \eta) f(\eta) d\eta, \quad \lambda > 0, \xi \in \mathcal{G}, f \in B(\mathcal{G}),$$

with

$$(2.4b) \quad r_\lambda^w(\xi, \eta) = r_\lambda(\xi, \eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi) S_{km}^w \frac{1}{\sqrt{2\lambda}} e_{\lambda,m}(\eta),$$

$$(2.4c) \quad S_{km}^w = 2w_m - \delta_{km},$$

where r_λ is defined in equation (1.24), and where $e_{\lambda,k}$, $e_{\lambda,m}$ denote the restrictions of e_λ (cf. (1.15)) to the edges l_k , l_m respectively.

Remark 2.4. The matrix $S^w = (S_{km}^w, k, m = 1, \dots, n)$ is the *scattering matrix* as defined in quantum mechanics. We briefly recall its construction in the present context, for more details the interested reader is referred to [28]. S^w is obtained from the boundary conditions at the vertex v in the following way. Consider a function f on \mathcal{G} which is continuously differentiable in $\mathcal{G}^\circ = \mathcal{G} \setminus \{v\}$, and such that for all $k = 1, \dots, n$ the limits

$$F_k = f(v_k) = \lim_{\xi \rightarrow v, \xi \in l_k^\circ} f(\xi)$$

$$F'_k = f'(v_k) = \lim_{\xi \rightarrow v, \xi \in l_k^\circ} f'(\xi)$$

exist. Define two column vectors $F, F' \in \mathbb{C}^n$, having the components F_k and F'_k , $k = 1, \dots, n$, respectively. Furthermore, consider boundary conditions of the following form

$$(2.5) \quad AF + BF' = 0,$$

where A and B are complex $n \times n$ matrices. The *on-shell scattering matrix at energy $E > 0$* is defined as

$$(2.6) \quad S_{A,B}(E) = -(A + i\sqrt{E}B)^{-1}(A - i\sqrt{E}B),$$

which exists and is unitary, provided the $n \times 2n$ matrix (A, B) has maximal rank (i.e., rank n) and AB^\dagger is hermitian. These requirements for A and B guarantee that the corresponding Laplace operator is self-adjoint on $L^2(\mathcal{G})$ (with Lebesgue measure). Observe that under these conditions the boundary conditions (2.5) are equivalent to any boundary conditions of the form $CAF + CBF' = 0$ where C is invertible. Also

$S_{CA,CB}(E) = S_{A,B}(E)$ holds true. For the Walsh process at hand, concrete choices for A and B are given by

$$A^w = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}, \quad B^w = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then (2.5) is the condition that f is actually continuous at the vertex v , i.e., $f(v_k) = f(v_m)$, $k, m = 1, \dots, n$, and that (2.2) is valid (with $w_k = b_k$, $k \in \{1, 2, \dots, n\}$). Obviously, (A^w, B^w) has maximal rank. However, $A^w(B^w)^\dagger$ is hermitian if and only if all b_k are equal (i.e., $b_k = 1/n$, $k = 1, \dots, n$). Nevertheless, (2.6) exists also in the non-hermitian case, and $S_{A^w, B^w}(E) = S^w$ holds for all $E > 0$ due to the relations $A^w S^w = -A^w$, and $B^w S^w = B^w$. In addition, the following relations are valid:

$$(2.7) \quad S^w = (S^w)^{-1},$$

$$(2.8) \quad \det S^w = (-1)^{n+1}.$$

Furthermore, S^w is a contraction, and the associated Laplace operator is m -dissipative on $L^2(\mathcal{G})$ since trivially $\text{Im}(AB^\dagger) = 0$, cf. Theorem 2.5 in [26]. When all b_k are equal, such that $A^w(B^w)^\dagger = 0$, then S^w is an involutive, orthogonal matrix of the form

$$(2.9) \quad S^w = -1 + 2P_n.$$

P_n is the matrix whose entries are equal to $1/n$. P_n is a real orthogonal projection, that is $P_n = P_n^\dagger = P_n^t = P_n^2$. It is also of rank 1, that is $\dim \text{Ran} P_n = 1$. The relation (2.4b) giving the resolvent in terms of the scattering matrix is actually valid in the more general context of arbitrary metric graphs and boundary conditions of the form (2.5), see [26, 29].

It is straightforward to compute the inverse Laplace transform of the right hand side of formula (2.4b), and this yields the following result.

Lemma 2.5. *For $t > 0$, $\xi, \eta \in \mathcal{G}$ the transition density of the Walsh process on \mathcal{G} is given by*

$$(2.10) \quad p^w(t, \xi, \eta) = p^D(t, \xi, \eta) + \sum_{k,m=1}^n 1_{l_k}(\xi) 2w_m g(t, d_v(\xi, \eta)) 1_{l_m}(\eta),$$

$$(2.11) \quad = p(t, \xi, \eta) + \sum_{k,m=1}^n 1_{l_k}(\xi) S_{km}^w g(t, d_v(\xi, \eta)) 1_{l_m}(\eta).$$

$p(t, \xi, \eta)$ is defined in equation (1.17), $p^D(t, \xi, \eta)$ in equation (1.21), g is the Gauß-kernel (1.4), and d_v is defined in equation (1.16).

Remark 2.6. Alternatively $p^w(t, \xi, \eta)$ can also be written as

$$(2.12) \quad p^w(t, \xi, \eta) = p(t, \xi, \eta) + \sum_{k,m=1}^n 1_{l_k}(\xi) \int_0^t P_\xi(H_v \in ds) S_{km}^w g(t-s, d(v, \eta)) 1_{l_m}(\eta).$$

Even though this formula appears somewhat more complicated than (2.11), it exhibits the role of the scattering matrix S^w , that is, it describes more clearly what happens when the process hits the vertex.

3. THE ELASTIC WALSH PROCESS

In this section we consider the boundary conditions (1.1) with $0 < a < 1$ and $c = 0$. The corresponding stochastic process, which we will denote by W^e , is constructed from the Walsh process W of the previous section in a similar way as the elastic Brownian motion on \mathbb{R}_+ is constructed from a reflected Brownian motion (cf., e.g., [19], [20, Chapter 2.3], [23, Chapter 6.2], [22, Chapter 6.4]).

In more detail, the construction is as follows. Consider the Walsh process W as discussed in the previous section. We may continue to suppose that W has been constructed pathwise from a standard Brownian motion B , as it has been described there. But then the local time of W at the vertex, denoted by L^w , is pathwise equal to the local time of the Brownian motion at the origin (and we continue to use the normalization determined by (1.8)). It is well-known (e.g., [20, 22, 23, 35]) that L^w has all properties of a PCHAF as formulated in subsection 1.5 for the construction of a subprocess by killing W at the vertex. We continue to denote the rate of the exponential random variable S used there by $\beta > 0$. Let W^e be the subprocess so obtained. In particular (cf. 1.5), W^e is a Brownian motion on \mathcal{G} , and in analogy with the case of a Brownian motion on the real line we call this stochastic process the *elastic Walsh process*. We write $\zeta_{\beta,0}$ for the lifetime of W^e (i.e., for the random time corresponding to ζ_β in subsection 1.5).

We proceed to show that the elastic Walsh process W^e has a generator A^e with domain $\mathcal{D}(A^e)$ which satisfies the boundary conditions as claimed. In other words, we claim that there exist $a \in (0, 1)$ and $b_k \in (0, 1)$, $k \in \{1, 2, \dots, n\}$, with $a + \sum_k b_k = 1$, so that for all $f \in \mathcal{D}(G)$,

$$(3.1) \quad af(v) = \sum_{k=1}^n b_k f'(v_k)$$

holds. To this end, we calculate $A^e f(v)$ in Dynkin's form. We shall use a notation similar to the one used in subsection 1.5. Namely, let \hat{P}_v and \hat{E}_v denote the probability and expectation, respectively, on the probability space extended by $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$, while the corresponding symbols without $\hat{\cdot}$ are those for the Walsh process without killing.

For $\epsilon > 0$ and under W^e let $H_{v,\epsilon}^e$ denote the hitting time of the complement $B_\epsilon(v)^c$ of the open ball $B_\epsilon(v)$ of radius $\epsilon > 0$ with center v . Then $H_{v,\epsilon}^e = H_{v,\epsilon}^w \wedge \zeta_{\beta,0}$,

where as before $H_{v,\epsilon}^w$ is the hitting time of $B_\epsilon(v)^c$ by the Walsh process W . (Note that $B_\epsilon(v)^c$ contains the cemetery Δ .) We find

$$(3.2) \quad \hat{E}_v \left(f(W^e(H_{v,\epsilon}^e)) \right) = \left(\sum_{k=1}^n w_k f_k(\epsilon) \right) \hat{P}_v(H_{v,\epsilon}^w < \zeta_{\beta,0}).$$

The probability in the last expression is taken care of by the following lemma.

Lemma 3.1. *For all $\epsilon, \beta > 0$,*

$$(3.3) \quad \hat{P}_v(H_{v,\epsilon}^w < \zeta_{\beta,0}) = \frac{1}{1 + \epsilon\beta}.$$

Proof. We may consider the Walsh process W as being pathwise constructed from a standard Brownian motion B on the real line as in the previous section, and we shall use the notations and conventions from there. Then it is clear that under P_v and under \hat{P}_v , $H_{v,\epsilon}^w$ has the same law as the hitting time of the point ϵ in \mathbb{R}_+ by the reflecting Brownian motion $|B|$ under Q_0 , that is, as $H_{\{-\epsilon,\epsilon\}}^B$ of the Brownian motion B itself under Q_0 . Let K^w denote the right continuous pseudo-inverse of L^w . For fixed $s \in \mathbb{R}_+$ we get

$$\{K_s^w < H_{v,\epsilon}^w\} = \{L^w(H_{v,\epsilon}^w) > s\}.$$

Hence

$$\begin{aligned} P_v(K_s^w < H_{v,\epsilon}^w) &= P_v(L^w(H_{v,\epsilon}^w) > s) \\ &= Q_0(L^B(H_{\{-\epsilon,\epsilon\}}^B) > s). \end{aligned}$$

In appendix B of [25] is shown with the method in [22, Section 6.4] that under Q_0 the random variable $L^B(H_{\{-\epsilon,\epsilon\}}^B)$ is exponentially distributed with mean ϵ . So we find

$$P_v(K_s^w < H_{v,\epsilon}^w) = e^{-s/\epsilon}.$$

We integrate this relation against the exponential law with rate β in the variable s , and obtain

$$\begin{aligned} \hat{P}_v(\zeta_{\beta,0} > H_{v,\epsilon}^w) &= 1 - \beta \int_0^\infty e^{-\beta s} P_v(K_s^w < H_{v,\epsilon}^w) ds \\ &= \frac{1}{1 + \epsilon\beta}. \end{aligned}$$

We used the fact that due to the continuity of the paths of W we have $\zeta_{\beta,0} \neq H_{v,\epsilon}^w$. \square

We insert formula (3.3) into equation (3.2), and obtain

$$\begin{aligned} A^e f(v) &= \lim_{\epsilon \downarrow 0} \frac{1}{\hat{E}_v(H_{v,\epsilon}^e)} \left(\hat{E}_v \left(f(W^e(H_{v,\epsilon}^e)) \right) - f(v) \right) \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\hat{E}_v(H_{v,\epsilon}^e)} \left(\frac{1}{1 + \epsilon\beta} \sum_{k=1}^n w_k f_k(\epsilon) - f(v) \right) \\ &= \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\hat{E}_v(H_{v,\epsilon}^e)} \frac{1}{1 + \epsilon\beta} \left(\sum_{k=1}^n w_k \frac{f_k(\epsilon) - f(v)}{\epsilon} - \beta f(v) \right). \end{aligned}$$

Obviously $\hat{E}_v(H_{v,\epsilon}^e) \leq E_v(H_{v,\epsilon}^w) = \epsilon^2$ (cf. lemma 2.1). Since the last limit and $f'(v_k)$, $k \in \{1, 2, \dots, n\}$, exist and are finite, we get as a necessary condition

$$(3.4) \quad \sum_{k=1}^n w_k f'(v_k) - \beta f(v) = 0.$$

Thus we have proved the following theorem.

Theorem 3.2. *Consider the boundary condition (1.1) with $a \in (0, 1)$, $b \in [0, 1]^n$, and $c = 0$. Set*

$$(3.5) \quad w_k = \frac{b_k}{1 - a}, \quad k = 1, \dots, n, \quad \beta = \frac{a}{1 - a},$$

and let W^e be the elastic Walsh process as constructed above with these parameters. Then the generator A^e of W^e is $1/2$ times the second derivative on \mathcal{G} with domain consisting of those $f \in C_0^2(\mathcal{G})$ which satisfy condition (1.1b).

Remark 3.3. Note that condition (1.1a) entails that if w_k and β are defined by (3.5) then $w_k \in [0, 1]$, $k = 1, \dots, n$, $\sum_k w_k = 1$, and $\beta > 0$. Therefore the choice (3.5) is consistent with the conditions on these parameters required by the construction of the elastic Walsh process W^e .

Next we compute the resolvent R^e of the elastic Walsh process. As a byproduct this will give another proof of theorem 3.2. Moreover, it will provide us with an explicit formula for the scattering matrix in this case. In contrast to the calculations in [20, Chapter 2.3], [23, Chapter 6.2] for the classical case with $\mathcal{G} = \mathbb{R}_+$, we do not use the first passage time formula (1.26), but instead we use formula (1.37). This simplifies the computation considerably.

Let $f \in C_0(\mathcal{G})$, $\lambda > 0$, and $\xi \in \mathcal{G}$. In the present context formula (1.37) reads

$$R_\lambda^e f(\xi) = R_\lambda^w f(\xi) - e_\lambda(\xi) \hat{E}_v(e^{-\lambda\zeta_{\beta,0}}) R_\lambda^w f(v),$$

where R^w is the resolvent of the Walsh process without killing, and e_λ is defined in (1.15). The Laplace transform of the density of $\zeta_{\beta,0}$ under \hat{P}_v is readily computed:

Lemma 3.4. *For all $\lambda, \beta > 0$,*

$$\hat{E}_v(e^{-\lambda\zeta_{\beta,0}}) = \frac{\beta}{\beta + \sqrt{2\lambda}}.$$

Proof. As remarked before, we may consider L^w to be equal to the local time at the origin of the Brownian motion B underlying the construction of W , and therefore the analogous statement is true for the right continuous pseudo-inverse K^w of L^w . As above let K^B denote the right continuous pseudo-inverse of L^B (cf. 1.2). Then for $s \in \mathbb{R}_+$,

$$\begin{aligned} E_v(e^{-\lambda K_s^w}) &= E_0^Q(e^{-\lambda K_s^B}) \\ &= e^{-\sqrt{2\lambda}s}, \end{aligned}$$

where we used lemma 1.7. Hence

$$\hat{E}_v(e^{-\lambda \zeta_{\beta,0}}) = \beta \int_0^\infty e^{-(\beta + \sqrt{2\lambda})t} dt,$$

which proves the lemma. \square

With lemma 3.4 we obtain the following formula

$$(3.6) \quad R_\lambda^e f(\xi) = R_\lambda^w f(\xi) - \frac{\beta}{\beta + \sqrt{2\lambda}} e_\lambda(\xi) R_\lambda^w f(v).$$

Note that $R_\lambda^w f$ is in the domain of the generator of the Walsh process, and therefore satisfies the boundary condition (2.2):

$$\sum_{k=1}^n w_k (R_\lambda^w f)'(v_k) = 0.$$

On the other hand, we obviously have $e'_\lambda(v_k) = -\sqrt{2\lambda}$ for all $k \in \{1, 2, \dots, n\}$. Thus with $\sum_{k=1}^n w_k = 1$ we find,

$$\sum_{k=1}^n w_k (R_\lambda^e f)'(v_k) = \beta \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}} R_\lambda^w f(v),$$

while equation (3.6) yields for $\xi = v$

$$R_\lambda^e f(v) = \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}} R_\lambda^w f(v).$$

The last two equations show that for all $f \in C_0(\mathcal{G})$, $\lambda > 0$, we have

$$\sum_{k=1}^n w_k (R_\lambda^e f)'(v_k) = \beta R_\lambda^e f(v).$$

Since for every $\lambda > 0$, R_λ^e maps $C_0(\mathcal{G})$ onto the domain of the generator of W^e , we have another proof of theorem 3.2.

Upon insertion of the expressions for the resolvent kernels of the Walsh process, equations (2.3), and (2.4), with the same notation as in lemma 2.3 we immediately obtain the following result:

Lemma 3.5. For $\lambda > 0$, $\xi, \eta \in \mathcal{G}$ the resolvent kernel of the elastic Walsh process W^e is given by

$$(3.7a) \quad r_\lambda^e(\xi, \eta) = r_\lambda^D(\xi, \eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi) 2w_m \frac{1}{\beta + \sqrt{2\lambda}} e_{\lambda,m}(\eta)$$

$$(3.7b) \quad = r_\lambda(\xi, \eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi) S_{km}^e(\lambda) \frac{1}{\sqrt{2\lambda}} e_{\lambda,m}(\eta),$$

with the scattering matrix S^e

$$(3.7c) \quad S_{km}^e(\lambda) = 2 \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}} w_m - \delta_{km}, \quad \lambda > 0, k, m \in \{1, 2, \dots, n\}.$$

Remark 3.6. Note that in contrast to the case of the Walsh process, this time the scattering matrix is not constant with respect to $\lambda > 0$. Also, when $\beta = 0$, formula (2.4c) is recovered, as it should be. In analogy with the discussion in remark 2.4, the boundary conditions for the elastic Walsh process is given by the matrices

$$A^e = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \beta \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}, \quad B^e = \begin{pmatrix} w_1 & w_2 & w_3 & \dots & w_{n-1} & w_n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

such that

$$\begin{aligned} S^e(\lambda) &= S_{A^e, B^e}(E = -2\lambda) \\ &= -(A^e + \sqrt{2\lambda} B^e)^{-1} (A^e - \sqrt{2\lambda} B^e). \end{aligned}$$

Observe that for $k, m \in \{1, \dots, n\}$ the matrix element $S_{km}^e(\lambda)$ of the scattering matrix is obtained from the resolvent kernel as

$$S_{km}^e(\lambda) = \sqrt{2\lambda} \lim_{\xi, \eta \rightarrow v} (r_\lambda^e(\xi, \eta) - r_\lambda(\xi, \eta)),$$

where the limit on the right hand side is taken in such a way that ξ, η converge to v along the edges l_k, l_m respectively. $S_{km}^e(\lambda)$ in turn fixes the data w_m and β , e.g., via the behavior for large λ , that is from the behavior at “large energies”

$$w_m = \frac{1}{2} (\delta_{km} + \lim_{\lambda' \uparrow \infty} S_{km}^e(\lambda')), \quad \text{for all } k \in \{1, 2, \dots, n\},$$

and

$$\beta = \sqrt{2\lambda} \left(\frac{\delta_{km} + \lim_{\lambda' \uparrow \infty} S_{km}^e(\lambda')}{\delta_{km} + S_{mm}^e(\lambda)} - 1 \right), \quad \text{for all } \lambda, \text{ and all } k, m \in \{1, 2, \dots, n\}.$$

Alternatively, the data can be obtained from the small λ behavior, that is the threshold behavior, of the scattering matrix, since from

$$\frac{w_m}{\beta} = \lim_{\lambda \downarrow 0} \frac{1}{2\sqrt{2\lambda}} (S_{km}^e(\lambda) + \delta_{km}) \quad \text{for all } k \in \{1, 2, \dots, n\},$$

we obtain

$$\beta^{-1} = \frac{1}{2\sqrt{2\lambda}} \left(\sum_{m=1}^n S_{km}^e(\lambda) + 1 \right) \quad \text{for all } k \in \{1, 2, \dots, n\},$$

and therefore

$$w_m = \frac{\lim_{\lambda \downarrow 0} \lambda^{-1/2} (S_{km}^e(\lambda) + \delta_{km})}{\lim_{\lambda \downarrow 0} \lambda^{-1/2} \left(\sum_m S_{k'm}^e(\lambda) + 1 \right)} \quad \text{for all } k, k' \in \{1, 2, \dots, n\}.$$

Furthermore we remark that in the context of quantum mechanics in the self-adjoint case $w_k = 1/n$, $k = 1, \dots, n$, the boundary conditions of the elastic Walsh process are interpreted as the presence of a δ -potential of strength β at the vertex.

In order to compute expressions for the transition kernel of the elastic Walsh process, we use the following two inverse Laplace transforms which follow from formulae (5.3.4) and (5.6.12) in [9] (cf. also appendix C in [25]) ($\lambda > 0$, $t \geq 0$, $x \geq 0$):

$$(3.8) \quad \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}} \xrightarrow{\mathcal{L}^{-1}} \epsilon_0(dt) - \beta \left(\frac{1}{\sqrt{2\pi t}} - \frac{\beta}{2} e^{\beta^2 t/2} \operatorname{erfc} \left(\beta \sqrt{\frac{t}{2}} \right) \right) dt,$$

$$(3.9) \quad \frac{1}{\beta + \sqrt{2\lambda}} e^{-\sqrt{2\lambda}x} \xrightarrow{\mathcal{L}^{-1}} g(t, x) - \frac{\beta}{2} e^{\beta x + \beta^2 t/2} \operatorname{erfc} \left(\frac{x}{\sqrt{2t}} + \beta \sqrt{\frac{t}{2}} \right).$$

Then the inverse Laplace transform of the scattering matrix S^e is given by the following measures on \mathbb{R}_+ :

$$(3.10) \quad \begin{aligned} s_{km}^e(dt) = & (2w_m - \delta_{km}) \epsilon_0(dt) - 2w_m \beta \frac{1}{\sqrt{2\pi t}} dt \\ & + w_m \beta^2 e^{\beta^2 t/2} \operatorname{erfc} \left(\beta \sqrt{\frac{t}{2}} \right) dt, \end{aligned}$$

with $k, m \in \{1, 2, \dots, n\}$. Moreover, for $t > 0$, $x \geq 0$, let us introduce

$$(3.11) \quad g_{\beta,0}(t, x) = g(t, x) - \frac{\beta}{2} e^{\beta x + \beta^2 t/2} \operatorname{erfc} \left(\frac{x}{\sqrt{2t}} + \beta \sqrt{\frac{t}{2}} \right).$$

Lemma 3.7. *For $t > 0$, $\xi, \eta \in \mathcal{G}$, the transition density p^e of the elastic Walsh process is given by*

$$(3.12) \quad p^e(t, \xi, \eta) = p^D(t, \xi, \eta) + \sum_{k,m=1}^n 1_{l_k}(\xi) 2w_m g_{\beta,0}(t, d_v(\xi, \eta)) 1_{l_m}(\eta),$$

and alternatively by

$$(3.13) \quad p^e(t, \xi, \eta) = p(t, \xi, \eta) + \sum_{k,m=1}^n 1_{l_k}(\xi) \left(\int_0^t P_\xi(H_v^w \in ds) \right. \\ \left. \times \left(s_{km}^e * g(\cdot, d(v, \eta)) \right)(t-s) \right) 1_{l_m}(\eta),$$

where $*$ denotes convolution.

4. THE WALSH PROCESS WITH A STICKY VERTEX

In this section we construct Brownian motions on \mathcal{G} with $a = 0$ in the boundary condition (1.1).

Consider the Walsh process W on \mathcal{G} from section 2 together with a right continuous, complete filtration \mathcal{F}^w , relative to which it is strongly Markovian. Furthermore, we denote its local time at the vertex v by L^w (cf. section 3).

Again we follow closely the recipe given by Itô and McKean in [19] (cf. also [23, Section 6.2]) for the case of a Brownian motion on the half line. For $\gamma \geq 0$ introduce a new time scale τ by

$$(4.1) \quad \tau^{-1} : t \mapsto t + \gamma L_t^w, \quad t \geq 0.$$

Since L^w is non-decreasing, τ^{-1} is strictly increasing. Moreover, we have $\tau^{-1}(0) = 0$ and $\lim_{t \rightarrow +\infty} \tau^{-1}(t) = +\infty$, which implies that τ exists, and is strictly increasing from \mathbb{R}_+ onto \mathbb{R}_+ , too. As is shown in [23, p. 160], the additivity of L^w entails the additivity of τ on its own time scale, i.e.:

Lemma 4.1. *For all $s, t \geq 0$, a.s. the following formula holds true*

$$(4.2) \quad \tau(s+t) = \tau(s) + \tau(t) \circ \theta_{\tau(s)}.$$

It is easily checked that for every $t \geq 0$, $\tau(t)$ is an \mathcal{F}^w -stopping time, and since τ is increasing, we obtain the subfiltration $\mathcal{F}^s = (\mathcal{F}_t^s, t \geq 0)$ of \mathcal{F}^w defined by $\mathcal{F}_t^s = \mathcal{F}_{\tau(t)}^w$, $t \in \mathbb{R}_+$. Moreover, we set $\mathcal{F}_\infty^w = \sigma(\mathcal{F}_t^w, t \in \mathbb{R}_+)$ and $\mathcal{F}_\infty^s = \sigma(\mathcal{F}_t^s, t \in \mathbb{R}_+)$, and find $\mathcal{F}_\infty^s \subset \mathcal{F}_\infty^w$. Standard calculations show that the completeness and the right continuity of \mathcal{F}^w entail the same properties for \mathcal{F}^s . (For details of the argument in the case where $\mathcal{G} = \mathbb{R}_+$ we refer the interested reader to section 3 of [25].)

Define a stochastic process W^s on \mathcal{G} , called *Walsh process with sticky vertex*, by

$$(4.3) \quad W_t^s = W_{\tau(t)}, \quad t \in \mathbb{R}_+.$$

Observe that when W is away from the vertex, L^w is constant, and therefore in this case τ^{-1} grows with rate 1. On the other hand, when W is at the vertex, τ^{-1} grows faster than with rate 1, and therefore τ increases slower than the deterministic time scale $t \mapsto t$. Thus W^s “experiences a slow down in time” until W has left the vertex. In this heuristic sense the vertex is “sticky” for W^s , because it spends more time there than W .

Note that because L^w has continuous paths, τ^{-1} and therefore also τ are pathwise continuous. Consequently, W^s has continuous sample paths. Since W has continuous paths, it is a measurable process, and hence for every $t \geq 0$, $W_{\tau(t)}$ is $\mathcal{F}_{\tau(t)}^w$ -measurable, that is, W^s is \mathcal{F}^s -adapted. Set $\theta_t^s = \theta_{\tau(t)}$. With the additivity (4.2) of τ we immediately find

$$(4.4) \quad W_s^s \circ \theta_t^s = W_{s+t}^s, \quad s, t \in \mathbb{R}_+.$$

Thus $\theta^s = (\theta_t^s, t \in \mathbb{R}_+)$ is a family of shift operators for W^s .

Next we show the strong Markov property of W^s relative to \mathcal{F}^s following the argument briefly sketched in section 6.2 of [23] for the case $\mathcal{G} = \mathbb{R}_+$. First we prove the simple Markov property of W^s with respect to \mathcal{F}^s . To this end, let $s, t \geq 0$, $\xi \in \mathcal{G}$, and $C \in \mathcal{B}(\mathcal{G})$. Then we get with (4.4)

$$\begin{aligned} P_\xi(W_{t+s}^s \in C \mid \mathcal{F}_t^s) &= P_\xi(W_s^s \circ \theta_t^s \in C \mid \mathcal{F}_t^s) \\ &= P_\xi(W_{\tau(s)} \circ \theta_{\tau(t)} \in C \mid \mathcal{F}_{\tau(t)}^w) \\ &= P_{W_{\tau(t)}}(W_{\tau(s)} \in C) \\ &= P_{W_t^s}(W_s^s \in C), \end{aligned}$$

where we used the strong Markov property of W with respect to \mathcal{F}^w . As a next step we prove that W^s has the strong Markov property for its hitting time H_v^s of the vertex. By construction, W^s and W have the same paths up to the hitting time of the vertex, and in particular H_v^s is also the hitting time of the vertex by W , that is, $H_v^s = H_v$. Moreover, since $L^w(H_v) = 0$, we get that $\tau^{-1}(H_v) = H_v = \tau(H_v)$, as well as $\theta^s(H_v) = \theta(H_v)$. Assume now that $t \geq 0$, $\xi \in \mathcal{G}$, and $C \in \mathcal{B}(\mathcal{G})$. Then on $\{H_v < +\infty\}$ we can compute with the strong Markov property of W as follows

$$\begin{aligned} P_\xi(W_{t+H_v}^s \in C \mid \mathcal{F}_{H_v}^w) &= P_\xi(W_t^s \circ \theta_{H_v}^s \in C \mid \mathcal{F}_{H_v}^w) \\ &= P_\xi(W_{\tau(t)} \circ \theta_{H_v} \in C \mid \mathcal{F}_{H_v}^w) \\ &= P_v(W_{\tau(t)} \in C) \\ &= P_v(W_t^s \in C). \end{aligned}$$

It is readily checked that $\mathcal{F}_{H_v}^s \subset \mathcal{F}_{H_v}^w$, and therefore we get in particular the strong Markov property of W^s with respect to $H_v^s = H_v$ in the form

$$(4.5) \quad P_\xi(W_{t+H_v}^s \in C \mid \mathcal{F}_{H_v}^s) = P_v(W_t^s \in C).$$

Finally, with the strong Markov property of the standard one-dimensional Brownian motions on every edge and the strong Markov property (4.5) just proved we can apply the arguments similar to those in [20, Section 3.6] to conclude that W^s is a Feller process. Hence it is strongly Markovian relative to the filtration \mathcal{F}^s .

By construction, W^s is up to time H_v^s equivalent to a standard one-dimensional Brownian motion, and it has continuous sample paths. Hence, altogether we have shown that W^s is a Brownian motion on \mathcal{G} in the sense of definition 1.1.

Now we want to compute the generator of W^s , and first we argue that v is not a trap for W^s . To this end, we may consider W as constructed from a standard

Brownian motion B as described in section 2. Let Z denote the zero set of B . Given $s \geq 0$ we can choose $t_0 \geq s$ in the complement Z^c of Z . Consider $t = \tau^{-1}(t_0)$, i.e., $t = t_0 + \gamma L_{t_0}^w$. Obviously $t \geq s$, and $\tau(t) \in Z^c$. Therefore $B_{\tau(t)} \neq 0$, and consequently $W_t^s = W_{\tau(t)} \neq v$.

Theorem 4.2. *Consider the boundary condition (1.1) with $a = 0$, $c \in (0, 1)$, and $b \in [0, 1]^n$. Set*

$$(4.6) \quad w_k = \frac{b_k}{1-c}, \quad k = 1, \dots, n, \quad \gamma = \frac{c}{1-c},$$

and let W^s be the sticky Walsh process as constructed above with these parameters. Then the generator A^s of W^s is $1/2$ times the second derivative on \mathcal{G} with domain consisting of those $f \in C_0^2(\mathcal{G})$ which satisfy condition (1.1b).

Before we prove theorem 4.2 we first prepare two preliminary results. Let $\epsilon > 0$, and let $H_{v,\epsilon}^s$ denote the hitting time of the complement of the open ball $B_\epsilon(v)$ with radius ϵ and center v by W^s . Recall that $H_{v,\epsilon}^w$ denotes the corresponding first hitting time for the Walsh process W .

Lemma 4.3. *P_v -a.s., the formula*

$$(4.7) \quad H_{v,\epsilon}^s = H_{v,\epsilon}^w + \gamma L_{H_{v,\epsilon}^w}^w$$

holds true.

Proof. Let W , and therefore also W^s , start in the vertex v . Since W^s and W have continuous paths with infinite lifetime we have for all $\gamma \geq 0$

$$H_{v,\epsilon}^s = \inf\{t > 0, d(v, W_{\tau(t)}) = \epsilon\},$$

and in particular for $\gamma = 0$,

$$H_{v,\epsilon}^w = \inf\{t > 0, d(v, W_t) = \epsilon\}.$$

Moreover, as argued above, both infima are a.s. finite. Set

$$\sigma = H_{v,\epsilon}^w + \gamma L_{H_{v,\epsilon}^w}^w.$$

Then $\tau(\sigma) = H_{v,\epsilon}^w$, and therefore

$$\begin{aligned} d(v, W_\sigma^s) &= d(v, W_{\tau(\sigma)}) \\ &= d(v, W_{H_{v,\epsilon}^w}) \\ &= \epsilon. \end{aligned}$$

Consequently we get $H_{v,\epsilon}^s \leq \sigma$. To derive the converse inequality we remark that

$$\begin{aligned} \epsilon &= d(v, W_{H_{v,\epsilon}^s}^s) \\ &= d(v, W_{\tau(H_{v,\epsilon}^s)}), \end{aligned}$$

which implies

$$\tau(H_{v,\epsilon}^s) \geq H_{v,\epsilon}^w.$$

Since τ is strictly increasing this entails

$$H_{v,\epsilon}^s \geq \tau^{-1}(H_{v,\epsilon}^w) = \sigma,$$

and the proof is finished. \square

Corollary 4.4. *For every $\gamma \geq 0$,*

$$(4.8) \quad E_v(H_{v,\epsilon}^s) = \epsilon^2 + \gamma\epsilon$$

holds.

Proof. By construction, the paths of W starting in v hit the complement of $B_\epsilon(v)$ exactly when the underlying standard Brownian motion B (cf. section 2) starting at the origin hits one of the points $\pm\epsilon$ on the real line. Thus under P_v , $L^w(H_{v,\epsilon}^w)$ has the same law as $L^B(H_{\{-\epsilon,\epsilon\}}^B)$ under P_0 . Lemma 1.8 states that under P_0 this random variable is exponentially distributed with mean ϵ . Then equation (4.8) follows directly from lemmas 4.3, and 2.1. \square

Given these results, we come to the

Proof of theorem 4.2. Let w_k , $k = 1, \dots, n$, and γ be defined as in (4.6), and note that due to the condition (1.1a) on b_k , $k = 1, \dots, n$, and c , we have $w_k \in [0, 1]$, $k = 1, \dots, n$, $\sum_k w_k = 1$, as well as $\gamma > 0$. Hence we can construct the associated sticky Walsh process W^s as above.

Let A^s denote the generator of W^s with domain $\mathcal{D}(A^s)$. Then we have for $f \in \mathcal{D}(A^s)$, $A^s f(v) = 1/2 f''(v)$ (cf. theorem 1.5). On the other hand, we can compute $A^s f(v)$ via Dynkin's formula as follows

$$\begin{aligned} A^s f(v) &= \lim_{\epsilon \downarrow 0} \frac{E_v(f(W^s(H_{v,\epsilon}^s))) - f(v)}{E_v(H_{v,\epsilon}^s)} \\ &= \lim_{\epsilon \downarrow 0} \frac{\sum_k w_k f_k(\epsilon) - f(v)}{\epsilon^2 + \gamma\epsilon}, \end{aligned}$$

where we used corollary 4.4. Since the directional derivatives of f at v

$$f'(v_k) = \lim_{\xi \rightarrow v, \xi \in l_k} \frac{f(\xi) - f(v)}{d(\xi, v)}, \quad k \in \{1, 2, \dots, n\},$$

exist (cf. lemma 1.3), we obviously get the boundary condition

$$(4.9) \quad \frac{1}{2} f''(v) = \frac{1}{\gamma} \sum_{k=1}^n w_k f'(v_k)$$

as a necessary condition. Finally, inserting of the values (4.6) of the parameters w_k , $k = 1, \dots, n$, and γ into equation (4.9) we complete the proof of theorem 4.2. \square

Next we shall compute the resolvent R^s of the Walsh process with sticky vertex. Similarly to the alternative proof of theorem 3.2 for the elastic Walsh process, as a byproduct we obtain an alternative proof of theorem 4.2. We begin with the following

Lemma 4.5. *Let $\lambda > 0$, $f \in C_0(\mathcal{G})$. Then*

$$(4.10) \quad \frac{1}{2} (R_\lambda^s f)''(v) = \frac{1}{\sqrt{2\lambda} + \gamma\lambda} \left(2\lambda (e_\lambda^w, f) - \sqrt{2\lambda} f(v) \right)$$

holds, where

$$(4.11) \quad e_\lambda^w(\xi) = w_k e_\lambda(\xi), \quad \xi \in l_k, k = 1, \dots, n,$$

and e_λ is defined in equation (1.15).

Proof. Let A^s be the generator of W^s on $C_0(\mathcal{G})$. From the identity $A^s R_\lambda^s = \lambda R_\lambda^s - \text{id}$, and the definition of τ we get

$$\begin{aligned} \frac{1}{2} (R_\lambda^s f)''(v) &= \lambda E_v \left(\int_0^\infty e^{-\lambda t} (f(W_t^s) - f(v)) dt \right) \\ &= \lambda E_v \left(\int_0^\infty e^{-\lambda(s+\gamma L_s^w)} (f(W_s) - f(v)) (ds + \gamma dL_s^w) \right) \\ &= \lambda E_v \left(\int_0^\infty e^{-\lambda(s+\gamma L_s^w)} (f(W_s) - f(v)) ds \right). \end{aligned}$$

In the last equality we used the fact that L^w only grows when W is at the vertex v . By construction of the Walsh process W we have

$$\begin{aligned} &E_v \left(e^{-\lambda\gamma L_s^w} (f(W_s) - f(v)) \right) \\ &= \sum_{k=1}^n w_k E_0 \left(e^{-\lambda\gamma L_s^B} (f_k(|B_s|) - f_k(0)) \right) \\ &= 2 \sum_{k=1}^n w_k \int_0^\infty \int_0^\infty e^{-\lambda\gamma y} (f_k(x) - f_k(0)) \frac{x+y}{\sqrt{2\pi s^3}} e^{-(x+y)^2/2s} dx dy, \end{aligned}$$

where we used lemma 1.6. We insert the last expression above, and use formula (1.31). This gives

$$\begin{aligned} \frac{1}{2} (R_\lambda^s f)''(v) &= 2\lambda \sum_{k=1}^n w_k \frac{1}{\sqrt{2\lambda} + \gamma\lambda} \int_0^\infty e^{-\sqrt{2\lambda}x} (f_k(x) - f_k(0)) dx \\ &= \frac{1}{\sqrt{2\lambda} + \gamma\lambda} \left(2\lambda (e_\lambda^w, f) - \sqrt{2\lambda} f(v) \right). \quad \square \end{aligned}$$

From the identity $A^s R_\lambda^s = \lambda R_\lambda^s - \text{id}$ and some simple algebra we get the

Corollary 4.6. *Let $\lambda > 0$, and $f \in C_0(\mathcal{G})$. Then*

$$(4.12) \quad R_\lambda^s f(v) = \frac{1}{\sqrt{2\lambda} + \gamma\lambda} \left(2(e_\lambda^w, f) + \gamma f(v) \right)$$

holds.

Since formula (1.27) in corollary 1.10 is valid for the resolvent of every Brownian motion on \mathcal{G} , we may use that formula for $R_\lambda^s f$, sum it against the weights w_k , $k = 1, \dots, n$, and insert the right hand side of equation (4.12). This results in

$$\sum_{k=1}^n w_k (R_\lambda^s f)'(v_k) = \gamma \frac{1}{\sqrt{2\lambda} + \gamma\lambda} (2\lambda (e_\lambda^w, f) - \sqrt{2\lambda} f(v)),$$

and a comparison with formula (4.10) shows that equation (4.9) holds true for f replaced by $R_\lambda^s f$ for arbitrary $f \in C_0(\mathcal{G})$. As promised we thus have another proof of theorem 4.2.

With the help of the first passage time formula we can now provide explicit expressions for the resolvent R^s , its kernel r^s and the transition kernel p^s of W^s . Inserting the right hand side of equation (4.12) into the first passage time formula (1.26), we immediately obtain for $f \in C_0(\mathcal{G})$, $\lambda > 0$,

$$(4.13) \quad R_\lambda^s f(\xi) = R_\lambda^D f(\xi) + \frac{1}{\sqrt{2\lambda} + \gamma\lambda} e_\lambda(\xi) (2(e_\lambda^w, f) + \gamma f(v)), \quad \xi \in \mathcal{G},$$

where R^D is the Dirichlet resolvent (1.22). Using formula (1.23) for the kernel of R^D together with (1.26), and (1.27), we get the following result.

Corollary 4.7. *For $\xi, \eta \in \mathcal{G}$, $\lambda > 0$, the resolvent kernel r_λ^s , of the Walsh process with sticky vertex is given by*

$$(4.14) \quad r_\lambda^s(\xi, d\eta) = r_\lambda^D(\xi, \eta) d\eta + \sum_{k,m=1}^n e_{\lambda,k}(\xi) 2w_m \frac{1}{\sqrt{2\lambda} + \gamma\lambda} e_{\lambda,m}(\eta) d\eta \\ + \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda} e_\lambda(\xi) \epsilon_v(d\eta),$$

with r_λ^D defined in (1.23), and ϵ_v denotes the Dirac measure in v . Alternatively, r_λ^s is given by

$$(4.15a) \quad r_\lambda^s(\xi, d\eta) = r_\lambda(\xi, \eta) d\eta + \sum_{k,m=1}^n e_{\lambda,k}(\xi) S_{km}^s(\lambda) \frac{1}{\sqrt{2\lambda}} e_{\lambda,m}(\eta) d\eta \\ + \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda} e_\lambda(\xi) \epsilon_v(d\eta),$$

where r_λ is defined in equation (1.24), and

$$(4.15b) \quad S_{km}^s(\lambda) = 2 \frac{\sqrt{2\lambda}}{\sqrt{2\lambda} + \gamma\lambda} w_m - \delta_{km}.$$

Remark 4.8. When all w_m , $m = 1, \dots, n$, are equal to $1/n$, the matrix $S^s(\lambda)$ takes the form

$$S^s(\lambda) = -1 + \frac{2\sqrt{2\lambda}}{\sqrt{2\lambda} + \gamma\lambda} P_n$$

which reduces to (2.9) when $\gamma = 0$. $S^s(\lambda)$ is unitary for all $\lambda < 0$. Also the $S^s(\lambda)$ for different λ all commute. As a consequence $S^s(\lambda)$ has the interpretation of a

quantum scattering matrix in the sense of [28]. More precisely, $S^s(\lambda)$ stems from the Schrödinger operator $-\Delta^s$, where Δ^s is a self-adjoint Laplace operator on $L^2(\mathcal{G})$ with boundary conditions of the form (2.5) with the choice

$$(4.16) \quad \begin{aligned} A &= -\frac{1}{2} (S^s(\lambda_0) + 1), \\ B &= -\frac{1}{2\sqrt{2\lambda_0}} (S^s(\lambda_0) - 1), \end{aligned}$$

for any λ_0 for which $\sqrt{2\lambda_0} + \gamma\lambda_0 \neq 0$. We emphasize that the Schrödinger operator $-\Delta^s$ and the generator A^s of the Walsh process are quite different: Not only do they act on different Banach spaces, but also the functions in the intersection of their domains satisfy different boundary conditions at the vertex v . As matter of fact, the integral kernel of the resolvent $(-\Delta^s + 2\lambda)^{-1}$ of the Schrödinger operator $-\Delta^s$ is given by, see Lemma 4.2 in [29],

$$\frac{1}{2} \left(r_\lambda(\xi, \eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi) S_{km}^s(\lambda) \frac{1}{\sqrt{2\lambda}} e_{\lambda,m}(\eta) \right),$$

that is — up to a factor 2 — by the right hand side of (4.15a) *without* the last term.

In more detail and with the definition (2.6)

$$S_{A,B}(E = -2\lambda) = S^s(\lambda)$$

holds for all $\lambda > 0$. As a function of k ($k^2 = E$), S^s is meromorphic in the complex k -plane with a pole on the positive imaginary axis at $k^b = 2i/\gamma$. This corresponds to a negative eigenvalue $E^b = -4/\gamma^2$ of $-\Delta^s$. The corresponding (normalized) eigenfunction ψ^b — physically speaking a *bound state* — is given as

$$\psi^b(\xi) = \frac{1}{2} \sqrt{\frac{\gamma}{n}} e^{-2d(v,\xi)/\gamma}, \quad \xi \in \mathcal{G}.$$

So quantum mechanically the vertex v acts like an attractive potential. We view this as a quantum analogue of the stickiness of the vertex v .

This analogy can be elaborated a bit further by inspecting the associated quantum mechanical time delay matrix (see, e.g., [1, 5, 21, 31–33, 39])

$$T(k) = \frac{1}{2ik} S(k)^{-1} \frac{\partial}{\partial k} S(k)$$

which in the present context gives

$$T(k) = \frac{-2\gamma}{k(4 + k^2\gamma^2)} P_n.$$

So $T(k)$ has zero as an $(n - 1)$ -fold eigenvalue plus the non-degenerate eigenvalue

$$\frac{-2\gamma}{k(4 + k^2\gamma^2)},$$

which for $\gamma > 0$ is the signal for a strict quantum delay. Observe that for $k \rightarrow +\infty$, that is for large energies, the time delay experienced by the quantum particle tends to

zero, while for $k \rightarrow 0$, i.e., for low energies, the delay becomes arbitrarily large. From the physical point of view, both effects are clearly to be expected. For comparison and in contrast to the present stochastic context, in quantum mechanics $\gamma < 0$ is also allowed for a meaningful Schrödinger operator and an associated scattering matrix.

Define for $x \geq 0$, $\gamma, t > 0$,

$$(4.17) \quad g_{0,\gamma}(t, x) = \frac{1}{\gamma} \exp\left(\frac{2x}{\gamma} + \frac{2t}{\gamma^2}\right) \operatorname{erfc}\left(\frac{x}{\sqrt{2t}} + \frac{\sqrt{2t}}{\gamma}\right).$$

It is not hard to check that

$$(4.18) \quad \lim_{\gamma \downarrow 0} g_{0,\gamma}(t, x) = g(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

Moreover, from [9, eq. (5.6.16)] (cf. also appendix C in [25]) the Laplace transform is

$$(4.19) \quad \mathcal{L}g_{0,\gamma}(\cdot, x)(\lambda) = \frac{1}{\sqrt{2\lambda} + \gamma\lambda} e^{-\sqrt{2\lambda}x}, \quad x \geq 0.$$

Observe that in agreement with (4.18)

$$\mathcal{L}g(\cdot, x)(\lambda) = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}x}$$

holds. Now we can readily compute the inverse Laplace transform of formulae (4.14), (4.15), and obtain the following result.

Corollary 4.9. *For $t > 0$, $\xi, \eta \in \mathcal{G}$ the transition kernel of the Walsh process with sticky vertex is given by*

$$(4.20) \quad \begin{aligned} p^s(t, \xi, \eta) &= p^D(t, \xi, \eta) d\eta \\ &+ \sum_{k,m=1}^n 1_{l_k}(\xi) 2w_m g_{0,\gamma}(t, d_v(\xi, \eta)) 1_{l_m}(\eta) d\eta \\ &+ \gamma g_{0,\gamma}(t, d(\xi, v)) \epsilon_v(d\eta) \end{aligned}$$

where p^D is defined in equation (1.21), or alternatively by

$$(4.21) \quad \begin{aligned} p^s(t, \xi, d\eta) &= p(t, \xi, \eta) d\eta \\ &+ \sum_{k,m=1}^n 1_{l_k}(\xi) \left(2w_m g_{0,\gamma}(t, d_v(\xi, \eta)) d\eta \right. \\ &\quad \left. - \delta_{km} g(t, d_v(\xi, \eta)) d\eta \right) 1_{l_m}(\eta) \\ &+ \gamma g_{0,\gamma}(t, d(\xi, v)) \epsilon_v(d\eta), \end{aligned}$$

and $p(t, a, b)$ is given in formula (1.17).

We close this section with some remarks concerning the local time of W^s at the vertex v , which also serve to prepare the construction of the most general Brownian motion on the single vertex graph \mathcal{G} in the next section.

Let us define

$$(4.22) \quad L_t^s = L_{\tau(t)}^w, \quad t \geq 0,$$

where — as before — L^w denotes the local time of the Walsh process at the vertex, having (cf. section 3) the same normalization as the local time of a standard one-dimensional Brownian motion (cf. (1.8)). By construction, L^s is pathwise continuous and non-decreasing. It is adapted to \mathcal{F}^s , and a straightforward calculation based on the additivity of L^w and formula (4.2) shows the (pathwise) additivity property

$$(4.23) \quad L_{s+t}^s = L_t^s + L_s^s \circ \theta_t^s, \quad s, t \geq 0.$$

Thus L^s is a PCHAF of (W^s, \mathcal{F}^s) . Furthermore, $t \geq 0$ is a point of increase for L^s if only if $\tau(t)$ is a point of increase for L^w , which only is the case if $W_{\tau(t)}$ is at the vertex, i.e., if W_t^s is at the vertex. Thus, it follows that L^s is a local time at the vertex for W^s . In order to completely identify it, it remains to compute its normalization, and it is not very hard to compute its α -potential (the interested reader can find the details for the case $\mathcal{G} = \mathbb{R}_+$ in [25]):

$$(4.24) \quad E_\xi \left(\int_0^\infty e^{-\alpha t} dL_t^s \right) = \frac{1}{\sqrt{2\alpha} + \gamma\alpha} e^{-\sqrt{2\alpha} d(\xi, v)}, \quad \alpha > 0, \xi \in \mathcal{G}.$$

5. THE GENERAL BROWNIAN MOTION ON A SINGLE VERTEX GRAPH

Finally, in this subsection we construct a Brownian motion W^g by killing the Walsh process with sticky vertex of section 4 in a similar way as in the construction of the elastic Walsh process (cf. section 3). W^g realizes the boundary condition (1.1) in its most general form.

Consider the sticky Walsh process W^s with stickiness parameter $\gamma > 0$, right continuous and complete filtration \mathcal{F}^s , and local time L^s at the vertex. We argued in section 4 that L^s is a PCHAF for (W^s, \mathcal{F}^s) , and therefore we can apply the method of killing described in section 1.5: We bring in the additional probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$ where P_β is the exponential law of rate $\beta > 0$, and the canonical coordinate random variable S . Then we take the family of product spaces $(\hat{\Omega}, \hat{\mathcal{A}}, (\hat{P}_\xi, \xi \in \mathcal{G}))$ of $(\Omega, \mathcal{A}, (P_\xi, \xi \in \mathcal{G}))$ and $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$. Define the random time

$$(5.1) \quad \zeta_{\beta, \gamma} = \inf \{ t \geq 0, L_t^s > S \}.$$

Then by the arguments given in section 1.5, the stochastic process W^g defined by $W_t^g = W_t^s$ for $t \in [0, \zeta_{\beta, \gamma})$, and $W_t^g = \Delta$ for $t \geq \zeta_{\beta, \gamma}$, is again a Brownian motion on \mathcal{G} in the sense of definition 1.1.

Denote by K^s the right continuous pseudo-inverse of L^s . Since L^s is continuous (cf. equation (4.22)), we get $L_{K_r^s}^s = r$ for all $r \in \mathbb{R}_+$. Recall that the right continuous pseudo-inverse of the local time L^w of the Walsh process was denoted by K^w . Then we have the following

Lemma 5.1. *For all $\gamma \geq 0$, the following relation holds true:*

$$(5.2) \quad K_r^s = K_r^w + \gamma r, \quad r \in \mathbb{R}_+.$$

Proof. For $\gamma, r \in \mathbb{R}_+$ define the random subset

$$J_\gamma(r) = \{t \geq 0, L_t^s > r\}$$

of \mathbb{R}_+ . Since L^s is pathwise increasing, this set is a random interval with endpoints K_r^s and $+\infty$. The relation $L_{K_r^s}^s = r$ implies that

$$J_\gamma(r) = (K_r^s, +\infty).$$

In particular, we have $J_0 = (K_r^w, +\infty)$. Now

$$t \in J_\gamma(r) \Leftrightarrow L_t^s = L_{\tau(t)}^w > r \Leftrightarrow \tau(t) \in J_0(r).$$

In other words, $J_\gamma(r) = \tau^{-1}(J_0(r))$, and therefore $K_r^s = \tau^{-1}(K_r^w)$ holds. From the definition of τ^{-1} (see equation (4.1)), and the relation $L_{K_r^s}^s = r$ we obtain formula (5.2) \square

In the proof of lemma 3.4 the Laplace transform of the density of $K_r^w, r \geq 0$, under P_v has been determined as $\lambda \mapsto \exp(-\sqrt{2\lambda}r)$. Hence we have

$$P_v(K_r^w \in dl) = \frac{r}{\sqrt{2\pi}l^3} e^{-r^2/2l} dl, \quad l \geq 0.$$

As a consequence we find the

Corollary 5.2. *For $r \geq 0$, K_r^s has the density*

$$(5.3) \quad P_v(K_r^s \in dl) = \frac{r}{\sqrt{2\pi}(l - \gamma r)^3} e^{-r^2/2(l - \gamma r)} dl, \quad l \geq \gamma r.$$

Furthermore, the Laplace transform of the density of K_r^s under P_v is given by

$$(5.4) \quad E_v(e^{-\lambda K_r^s}) = e^{-(\sqrt{2\lambda} + \lambda\gamma)r}, \quad \lambda > 0.$$

Remark 5.3. One can use lemma C.1 in [25] to check that the right hand side of equation (5.3) is indeed the inverse Laplace transform of the right hand side of formula (5.4).

Observe that $\zeta_{\beta,\gamma} = K_S^s$ and $\zeta_{\beta,0} = K_S^w$. Thus we obtain the

Corollary 5.4. *For all $\beta > 0, \gamma \geq 0$, the following equation holds true*

$$(5.5) \quad \zeta_{\beta,\gamma} = \zeta_{\beta,0} + \gamma S.$$

As before, \hat{E}_ξ denotes the expectation with respect to $\hat{P}_\xi, \xi \in \mathcal{G}$.

Corollary 5.5. *For all $\beta > 0, \gamma \geq 0, \lambda > 0$, the following formula holds true*

$$(5.6a) \quad \hat{E}_v(e^{-\lambda \zeta_{\beta,\gamma}}) = \beta \rho(\lambda),$$

with

$$(5.6b) \quad \rho(\lambda) = \frac{1}{\beta + \sqrt{2\lambda} + \gamma\lambda}.$$

Proof. With corollary 5.2 and $\zeta_{\beta,\gamma} = K_S^s$ we obtain

$$\begin{aligned}\hat{E}_v(e^{-\lambda\zeta_{\beta,\gamma}}) &= \beta \int_0^\infty E_v(e^{-\lambda K_r^s}) e^{-\beta r} dr \\ &= \frac{\beta}{\beta + \sqrt{2\lambda} + \gamma\lambda}.\end{aligned}\quad \square$$

Denote by R^g the resolvent of W^g . With lemma 1.11 we immediately find the

Corollary 5.6. *For all $f \in C_0(\mathcal{G})$, $\lambda > 0$, $\xi \in \mathcal{G}$ the following formula holds true:*

$$(5.7) \quad R_\lambda^g f(\xi) = R_\lambda^s f(\xi) - \beta \rho(\lambda) e_\lambda(\xi) R_\lambda^s f(v).$$

Now it is easy to verify that for appropriately chosen parameters $\beta, \gamma, w_k, k = 1, \dots, n$, the Brownian motion W_g realizes the boundary condition (1.1b).

Theorem 5.7. *Consider the boundary condition (1.1), and assume that b is not the null vector. Set $r = a + c \in (0, 1)$, and*

$$(5.8) \quad w_k = \frac{b_k}{1-r}, \quad k = 1, \dots, n, \quad \beta = \frac{a}{1-r}, \quad \gamma = \frac{c}{1-r}.$$

Let W^g be the Brownian motion as constructed above with these parameters. Then the generator A^g of W^g is $1/2$ times the Laplace operator on \mathcal{G} with domain $\mathcal{D}(A^g)$ consisting of those $f \in C_0^2(\mathcal{G})$ which satisfy condition (1.1b).

Proof. As in the previous cases it is readily seen that the definition (5.8) of the parameters $\gamma, \beta, w_k, k = 1, \dots, n$, is consistent with the conditions used in the above construction of W^g .

Let A^g be the generator of W^g with domain $\mathcal{D}(A^g)$. Since W^g is a Brownian motion on \mathcal{G} in the sense of definition 1.1, it follows from theorem 1.5 that $\mathcal{D}(A^g) \subset C_0^2(\mathcal{G})$, and that for all $f \in \mathcal{D}(A^g)$, $A^g f(\xi) = 1/2 f''(\xi)$, $\xi \in \mathcal{G}$. Let $h \in C_0(\mathcal{G})$, $\lambda > 0$. Then $R_\lambda^g h \in \mathcal{D}(A^g)$, and therefore we may compute with equation (5.7) as follows

$$\begin{aligned}\frac{\gamma}{2} (R_\lambda^g h)''(v) &= \frac{\gamma}{2} (R_\lambda^s h)''(v) - \beta \rho(\lambda) 2\lambda (R_\lambda^s h)(v) \\ &= \sum_{k=1}^n w_k (R_\lambda^s h)'(v_k) - \beta \rho(\lambda) \gamma \lambda (R_\lambda^s h)(v),\end{aligned}$$

where we used the fact that, since $R_\lambda^s h$ is in the domain of the generator A^s of W^s , it satisfies the boundary condition (4.9). We rewrite this equation in the following way:

$$(5.9) \quad \begin{aligned}\frac{\gamma}{2} (R_\lambda^g h)''(v) &= \sum_{k=1}^n w_k (R_\lambda^s h)'(v_k) + \beta \sqrt{2\lambda} \rho(\lambda) (R_\lambda^s h)(v) \\ &\quad - \beta \rho(\lambda) (\sqrt{2\lambda} + \gamma \lambda) (R_\lambda^s h)(v).\end{aligned}$$

Now we differentiate equation (5.7) at $\xi \in l_k$, $k = 1, \dots, n$, let ξ tend to v along any edge l_k , and sum the resulting equation against the weights w_k , $k = 1, \dots, n$. Then we get the following formula

$$(5.10) \quad \sum_{k=1}^n w_k (R_\lambda^g h)'(v_k) = \sum_{k=1}^n w_k (R_\lambda^s h)'(v_k) + \beta \sqrt{2\lambda} \rho(\lambda) (R_\lambda^s h)(v),$$

where we used $\sum_k w_k = 1$. On the other hand, for $\xi = v$, equation (5.7) gives

$$(5.11) \quad (R_\lambda^g h)(v) = \rho(\lambda) (\sqrt{2\lambda} + \gamma \lambda) (R_\lambda^s h)(v).$$

A comparison of equations (5.10), (5.11) with (5.9) shows that we have proved the following formula

$$(5.12) \quad \frac{\gamma}{2} (R_\lambda^g h)''(v) = \sum_{k=1}^n w_k (R_\lambda^g h)'(v_k) - \beta (R_\lambda^g h)(v).$$

With the values (5.8) for β , γ , and w_k , $k = 1, \dots, n$, it is obvious that $f = R_\lambda^g h$ satisfies equation (1.1b). Since R_λ^g is surjective from $C_0(\mathcal{G})$ onto the domain of the generator A^g of W^g , the proof of the theorem is finished. \square

Let $\lambda > 0$, $f \in C_0(\mathcal{G})$. Insertion of the right hand side of formula (4.13) for R_λ^s into equation (5.7) gives us after some simple algebra the following expression for $R_\lambda^g f$:

$$(5.13) \quad R_\lambda^g f(\xi) = R_\lambda^D f(\xi) + \rho(\lambda) e_\lambda(\xi) (2(e_\lambda^w, f) + \gamma f(v)), \quad \xi \in \mathcal{G},$$

where R_λ^D is the Dirichlet resolvent, e_λ is defined in equation (1.15), e_λ^w in equation (4.11), and $\rho(\lambda)$ is as in formula (5.6b). From equation (5.13) we can read off the following result:

Corollary 5.8. *For $\xi, \eta \in \mathcal{G}$, $\lambda > 0$, the resolvent kernel r_λ^g of the general Brownian motion W^g on \mathcal{G} is given by*

$$(5.14) \quad r_\lambda^g(\xi, d\eta) = r_\lambda^D(\xi, \eta) d\eta + \sum_{k,m=1}^n e_{\lambda,k}(\xi) 2w_m \rho(\lambda) e_{\lambda,m}(\eta) d\eta \\ + \gamma \rho(\lambda) e_\lambda(\xi) \epsilon_v(d\eta),$$

with r_λ^D as in formula (1.23), and ρ is defined in equation (5.6b). Alternatively, r_λ^g can be written in the following form

$$(5.15a) \quad r_\lambda^g(\xi, d\eta) = r_\lambda(\xi, \eta) d\eta + \sum_{k,m=1}^n e_{\lambda,k}(\xi) S_{km}^g(\lambda) \frac{1}{\sqrt{2\lambda}} e_{\lambda,m}(\eta) d\eta \\ + \gamma \rho(\lambda) e_\lambda(\xi) \epsilon_v(d\eta),$$

where r_λ is defined in equation (1.24), and

$$(5.15b) \quad S_{km}^g(\lambda) = 2\sqrt{2\lambda} \rho(\lambda) w_m - \delta_{km}.$$

In order to invert the Laplace transforms in equations (5.14), (5.15), we define for $\beta, \gamma > 0$, the following function $g_{\beta, \gamma}$ on $(0, +\infty) \times \mathbb{R}_+$:

$$(5.16) \quad g_{\beta, \gamma}(t, x) = \frac{1}{\gamma^2} \frac{1}{\sqrt{2\pi}} \int_0^t \frac{s + \gamma x}{(t-s)^{3/2}} \exp\left(-\frac{(s + \gamma x)^2}{2\gamma^2(t-s)}\right) e^{-\beta s/\gamma} ds,$$

with $(t, x) \in (0, +\infty) \times \mathbb{R}_+$. The heat kernel $g_{\beta, \gamma}$ is discussed in more detail in appendix C of [25]. In particular, it is outlined there that the limits of $g_{\beta, \gamma}$ as $\beta \downarrow 0$, and $\gamma \downarrow 0$, yield the kernels $g_{\beta, 0}$ (equation (3.11)) and $g_{0, \gamma}$ (equation (4.17)), respectively. Moreover, it is proved there that the Laplace transform of $g_{\beta, \gamma}(\cdot, x)$, $x \geq 0$, is given by

$$(5.17) \quad \rho(\lambda) e^{-\sqrt{2\lambda}x}, \quad \lambda > 0,$$

where ρ is defined in (5.6b). Hence we get the

Corollary 5.9. *For $\xi, \eta \in \mathcal{G}$, $t > 0$, the transition kernel of the general Brownian motion W^g on \mathcal{G} is given by*

$$(5.18) \quad \begin{aligned} p^g(t, \xi, d\eta) &= p^D(t, \xi, \eta) d\eta \\ &+ \sum_{k,m=1}^n 1_{l_k}(\xi) 2w_m g_{\beta, \gamma}(t, d_v(\xi, \eta)) 1_{l_m}(\eta) d\eta \\ &+ \gamma g_{\beta, \gamma}(t, d(\xi, v)) \epsilon_v(d\eta), \end{aligned}$$

which alternatively can be written as

$$(5.19) \quad \begin{aligned} p^g(t, \xi, d\eta) &= p(t, \xi, \eta) d\eta \\ &+ \sum_{k,m=1}^n 1_{l_k}(\xi) \left(2w_m g_{\beta, \gamma}(t, d_v(\xi, \eta)) \right. \\ &\quad \left. - \delta_{km} g(t, d_v(\xi, \eta)) \right) 1_{l_m}(\eta) d\eta \\ &+ \gamma g_{\beta, \gamma}(t, d(\xi, v)) \epsilon_v(d\eta). \end{aligned}$$

REFERENCES

- [1] W. Amrein, J.M. Jauch, and K. Sinha, *Scattering Theory in Quantum Mechanics*, Benjamin, Reading Mass., 1977.
- [2] M. Barlow, J. Pitman, and M. Yor, *On Walsh's Brownian motion*, Séminaire de Probabilités XXIII (J. Azéma, P. A. Meyer, and M. Yor, eds.), Lecture Notes in Mathematics, no. 1372, Springer-Verlag, Berlin, Heidelberg, New York, 1989, pp. 275–293.
- [3] R. Blumenthal, *An extended Markov property*, Trans. American Math. Soc. **85** (1957), 52–72.
- [4] R. M. Blumenthal and R. K. Gettoor, *Markov Processes and Potential Theory*, Academic Press, New York and London, 1968.
- [5] R. Brunetti and K. Fredenhagen, *Time of occurrence observables in quantum mechanics*, Phys. Rev. A **66** (2002), 044101–1–044101–3.
- [6] E. B. Dynkin, *Die Grundlagen der Theorie der Markoffschen Prozesse*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1961.
- [7] E.B. Dynkin, *Markov Processes*, vol. 1, Springer-Verlag, Berlin, Heidelberg, New York, 1965.
- [8] ———, *Markov Processes*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, 1965.

- [9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, vol. I, McGraw-Hill, New York, Toronto, London, 1954.
- [10] P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplyaev (eds.), *Analysis on Graphs and Its Applications*, Proc. Symp. Pure Math., vol. 77, Providence, American Math. Soc., 2010.
- [11] W. Feller, *The parabolic differential equations and the associated semi-groups of transformations*, Ann. Math. **3** (1952), 468–519.
- [12] ———, *Diffusion processes in one dimension*, Trans. American Math. Soc. **77** (1954), 1–31.
- [13] ———, *The general diffusion operator and positivity preserving semi-groups in one dimension*, Ann. Math. **60** (1954), 417–436.
- [14] ———, *An Introduction to Probability Theory and Its Applications*, 2nd ed., vol. 2, Wiley, New York, London, Sydney, 1971.
- [15] T. Hida, *Brownian Motion*, Applications of Mathematics, no. 11, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
- [16] G. A. Hunt, *Some theorems concerning Brownian motion*, Trans. American Math. Soc. **81** (1956), 294–319.
- [17] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North Holland, Amsterdam, Oxford, New York, 1989.
- [18] K. Itô, *Poisson point processes attached to Markov processes*, Proc. Sixth Berkeley Symp. Math. Statist. Probab., vol. 3, University of California Press, 1972.
- [19] K. Itô and H. P. McKean Jr., *Brownian motions on a half line*, Illinois J. Math. **7** (1963), 181–231.
- [20] ———, *Diffusion Processes and their Sample Paths*, 2nd ed., Springer-Verlag, Berlin, Heidelberg, New York, 1974.
- [21] J.M. Jauch, B. Misra, and K.B. Sinha, *Time delay in scattering processes*, Helv. Phys. Acta **45** (1972), 398–426.
- [22] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer-Verlag, Berlin, Heidelberg, New York, 1991.
- [23] F.B. Knight, *Essentials of Brownian Motion and Diffusion*, Mathematical Surveys and Monographs, vol. 18, American Mathematical Society, Providence, Rhode Island, 1981.
- [24] V. Kostykin, J. Potthoff, and R. Schrader, *Brownian Motions on Metric Graphs*, to appear.
- [25] V. Kostykin, J. Potthoff, and R. Schrader, *Brownian motions on metric graphs: Feller Brownian motions on intervals revisited*, arXiv: 1008.3761, August 2010.
- [26] ———, *Contraction Semigroups on Metric Graphs*, Analysis on Graphs and Its Applications (Providence) (P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplyaev, eds.), Proc. Symp. Pure Math., vol. 77, Amer. Math. Soc., 2010, pp. 423–458.
- [27] V. Kostykin, J. Potthoff, and R. Schrader, *A Note on Feller Semigroups and Resolvents*, arXiv: 1102.3979, February 2011.
- [28] V. Kostykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A: Math. Gen. **32** (1999), 595–630.
- [29] ———, *Laplacians on metric graphs: Eigenvalues, resolvents and semigroups*, Quantum Graphs and Their Applications (Providence, RI) (G. Berkolaiko, R. Carlson, S.A. Fulling, and P. Kuchment, eds.), Contemp. Math., vol. 415, Amer. Math. Soc., 2006.
- [30] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, 2nd ed., Gauthier-Villars, Paris, 1965.
- [31] Ph. A. Martin, *On the time delay of simple scattering systems*, Commun. Math. Phys. **47** (1976), 221–227.
- [32] ———, *Time delay in quantum scattering processes*, New developments in mathematical physics, Schlading, 1981, vol. XXIII, 1981, pp. 157–208.
- [33] H. Narnhofer, *Another definition of time delay*, Phys.Rev.D **22** (1980), 2387–2390.
- [34] D. Ray, *Stationary Markov processes with continuous paths*, Trans. American Math. Soc. **82** (1956), 452–493.
- [35] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

- [36] Th. S. Salisbury, *Construction of right processes from excursions*, Probab. Th. Rel. Fields **73** (1986), 351–367.
- [37] ———, *On the Itô excursion process*, Probab. Th. Rel. Fields **73** (1986), 319–350.
- [38] J.B. Walsh, *A diffusion with a discontinuous local time*, Astérisque **52–53** (1978), 37–45.
- [39] E. P. Wigner, *Lower limit for the energy derivative of the scattering phase shift*, Phys. Rev. **98** (1955), 145–147.
- [40] D. Williams, *Diffusions, Markov Processes, and Martingales*, John Wiley & Sons, Chichester, New York, Brisbane, Toronto, 1979.

VADIM KOSTRYKIN
INSTITUT FÜR MATHEMATIK
JOHANNES GUTENBERG–UNIVERSITÄT
D–55099 MAINZ, GERMANY
E-mail address: kostrykin@mathematik.uni-mainz.de

JÜRGEN POTTHOFF
INSTITUT FÜR MATHEMATIK, UNIVERSITÄT MANNHEIM
D–68131 MANNHEIM, GERMANY
E-mail address: potthoff@math.uni-mannheim.de

ROBERT SCHRADER
INSTITUT FÜR THEORETISCHE PHYSIK
FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 14
D–14195 BERLIN, GERMANY
E-mail address: schrader@physik.fu-berlin.de